A short proof of anti-Ramsey number for cycles

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Abstract
Ramsey's theorem states that there exists a least positive integer R(r, s) for which every blue-red edge colouring of the complete graph on R(r, s) vertices contains a blue clique on r vertices or a red clique on s vertices. This work contains a simplified proof of Anti-Ramsey theorem for cycles. If there is an edge e between H and H0, incident to, say, some v ∈ H of color from NEWc(v), then we can make H and H0 connected by adding the edge e and deleting some edge incident to v of the same color as e in H, so the resulting graph G’ has a connected component of order ≥ 2(k+1/2), which contradicts that every connected component is of order ≤ k−1. Since each component is Hamiltonian and of order ≥ k+1/2, to avoid a rainbow Ck, by the same type of argument as in Claim 1, we must have that |c(H, H0)| = 1.

Keywords: Ramsey’s theorem, anti-ramsey, edge colouring

Introduction
For a graph H, the classical anti-Ramsey number AR(n, H) is the maximum number of colors in a coloring of edges of Kn with no rainbow copy of H. It was introduced by Erdos, Simonovits and Sós [1975]. When H is a cycle of length k, Ck, they provided a rainbow Ck-free coloring of edges of Kn with n ⌈k−2/2 + 1/(k−1)⌉ + O(1) colors and conjectured that this is optimal. They proved the conjecture for k = 3. Alon [1] proved the conjecture for k = 4 and derived an upper bound for general k. In [4], Jiang and West improved the general upper bound and mentioned that the conjecture has been proven for k ≤ 7. Finally, Montellano-Ballesteros and Neumann-Lara (2005) proved the conjecture completely. The main technique used in the previous study is a careful, detailed analysis of a graph representing the coloring, in particular, proving that each component of such a graph is Hamiltonian if each vertex has enough “new” colors. This paper uses the same idea as in earlier study, but shortens the proof.

Theorem 1. For k ≥ 3 and n ∈ N, AR(n, Ck) ≤ n ⌈k−2/2 + 1/(k−1)⌉ − 1

Definitions and proofs of main results
Let K = Kn for a fixed n. For an edge coloring c of K, and a vertex v ∈ V (K), let the set of new colors at v, NEWc(v), be the set of colors used on edges between v and V (K) \ {v}, but not used on edges spanned by V (K) \ {v}. Let newc(v) = |NEWc(v)|. Then the number of colors used by c on K, |c|, equals newc(v) + |c(K \ v)|, where for a subgraph H of K, |c(H)| denotes the number of colors used by c on the edges of H. Here we simply have written |c| instead of |c(K)|. For pairwise disjoint subsets X, Y, Z of V (K), let K[X] be the subgraph induced by X, K[X, Y ] the bipartite subgraph induced by X and Y , K[X, Y, Z] the tripartite subgraph induced by X, Y , and Z. Then the corresponding sets of colors used in those subgraphs are denoted by c(X), c(X, Y ), and c(X, Y, Z) respectively. For a subgraph H of a graph G and a vertex v of G, let degH(v) := |NG(v) ∩ V (H)|. We now state a version of the Dirac and Ore’s theorems for Hamiltonian cycle which is essential for our proofs.

Theorem 2 (Dirac, 1952; Ore, 1960). Let P = v1, v2, . . . , vm, m ≥ 3, be a path in a connected graph G. Suppose degP (v1) + degP (vm) ≥ m.

1. Then V (P) contains a cycle of length m in G.
2. If P is a longest path in G, then G is Hamiltonian.

We define a few special edge colorings of a complete graph with no rainbow Ck. We say that an edge-coloring c of K is weak k-anticyclic if there is a partition of V (K) into sets V1, . . . , Vt with 1 ≤ |Vi | ≤ k−1, i = 1, . . . , t, such that (i) for any i, j with...
1 ≤ i < j ≤ t, \( |c(V_i, V_j)| = 1 \); (ii) for any i, j, ` with 1 ≤ i < j ≤ t, \( |c(V_i, V_j, V')| ≤ 2 \); and (iii) c has no rainbow C_k.

In addition, if all but at most one of the parts of the partition are exactly of size \( k - 1 \) and the edges spanned by each Vi have own colors (i.e., colors used only once), then c is called k-anticyclic.

We note a fixed coloring from the set of k-anticyclic colorings of \( K_n \) such that the color of any edge between Vi and Vj is \( \min(i, j) \), by c*. Then we easily see the following. Lemma 2.2. If c is weak k-anticyclic, then

\[ |c| ≤ |c*| ≤ n \{ k - 2/2 + 1/k - 1 \} - 1. \]

Next lemma is the main tool for the proof of the main theorem. It appears in a different form in [Montellano-Ballesteros and Neumann-Lara (2005), Lemma 9]. We include it here for completeness.

Lemma 2.2. Let \( k ≥ 4 \). Let c be an edge-coloring of \( K \) with no rainbow C_k. If for all \( x, y \in V(K) \) with \( x ≠ y \), \( \text{newc}(x) ≥ 2 \) and \( \text{newc}(x) + \text{newc}(y) ≥ k - 1 \) (3.1) then c is weak k-anticyclic. Proof. Consider a representing graph G of c such that it spans \( K \) and has exactly one edge of each color from \( \{\text{newc}(v) \mid v \in V(K)\} \). The hypothesis (3.1) gives a bound on degrees of vertices in G, namely the sum of degrees of two distinct vertices in G is at least \( k - 1 \). In the following, H denotes a connected component of G.

Claim 1 If there is a cycle of length \( k - 1 \) in H, then \( |V(H)| = k - 1 \). (1.1)

Suppose not, i.e., there is a cycle, \( (v_1, \ldots, v_k) \), and \( V(H) \setminus \{v_1, \ldots, v_k\} = \emptyset \). Since H is connected, some \( u \in V(H) \setminus \{v_1, \ldots, v_k\} \) is adjacent to some vertex in \( \{v_1, \ldots, v_k\} \), say v1. If \( (c(u, v_1) \in \text{NEWc}(v_1) \), then \( c(u, v_2) = c(v_2, v_3) \); otherwise \( c(u, v_1, v_2, v_3, \ldots, v_k, v_1) \) in K is a rainbow C_k. Similarly \( c(u, v_3) = c(v_3, v_4), \ldots, c(u, v_k) = c(v_k, v_1) \), and eventually \( c(u, v_1) = c(v_1, v_2) \), which contradicts that vv1 and vv2 are edges of H. Hence c(u, v1) \in \text{NEWc}(u). By the similar argument as above, we have c(u, \{v_1, \ldots, v_k\}) = \emptyset \in \text{NEWc}(u). Since we assumed newc(u) ≥ 2, there is \( w \in V(H) \{v_1, \ldots, v_k\} \) with c(u, w) \in \text{NEWc}(u) and c(u, w) = \emptyset \in \text{NEWc}(u). Considering cycles of length \( k \) in K, (vk−2, u, w, v1, v2, \ldots, v_k) and (v1, w, u, v_2, v_3, \ldots, v_k−1, v1), we have c(w, v1) = c(v_1, v_2) = c(v_k−1, v_1), which contradicts that v1v2 and v1−1v1 are edges of H. Claim 2 \( k + 1/2 ≤ |V(H)| ≤ k - 1 \) (Hence H is Hamiltonian from by (1.1) and Theorem 1)

The lower bound follows from (3.1). If in H every path has at most \( k - 1 \) vertices or there is a cycle of length \( k - 1 \), then from Theorem 1 and Claim 1, we have that the upper bound holds. Hence we may assume that in H there is a path on at least k vertices, but no C_k. In particular, we can find a path, \( v_1, \ldots, v_k \), satisfying \( c(v_k−1, v_k) \in \text{NEWc}(v_k−1) \) since (i) considering \( P_1 := v_1, \ldots, v_k \), to avoid C_k, we have degP_1(v1) + degP_1(vk−1) < k − 1; (ii) from (1.1), without loss of generality we can find a \( k \in V(H) \setminus V(P_1) \) such that \( v_k−1v_k \) is an edge of H and \( c(v_k−1, v_k) \in \text{NEWc}(v_k−1) \). Let \( P_2 := v_2, \ldots, v_k \). Then degP_2(v2) ≥ newc(v2), (3.2) since otherwise there is \( x \in V(H) \setminus V(P_2) \) such that \( c(x, v_2) \in \text{NEWc}(v_2) \) and \( c(x, v_2) = c(v_2, v_3) \), in which case we obtain a rainbow C_k in K, namely \( x, v_2, v_k \). Also we have degP_2(vk) < newc(vk) since otherwise together with (3.2), V (P2) induces a cycle of length \( k - 1 \) in H by Theorem 1. Therefore we can find a \( v_k+1 \in V(H) \setminus V(P_2) \) such that \( v_kv_k+1 \) is an edge of H and \( c(v_k, v_k+1) \in \text{NEWc}(v_k) \). Note that \( v_k+1 ≠ v_2 \) since otherwise \( v_2, v_k, v_k+1 \) is a rainbow C_k−1 in H. Let \( P_3 := \{v_3, \ldots, v_k, v_k+1\} \). Then degP_3(v3) ≥ newc(v3), (3.3) since otherwise there is \( y \in V(H) \setminus V(P_3) \) such that \( c(y, v_3) \in \text{NEWc}(v_3) \) and \( c(y, v_3) = c(v_3, v_k) \), so \( (v_3, v_k, v_k+1) \) is a rainbow C_k in K. Now we note that \( c(v_2, v_k+1) = c(v_2, v_3) \) to avoid a rainbow C_k induced by \( \{v_2, v_k+1, v_3\} \) in K. Let \( S = \{1 + v_2, v_3, \ldots, k - 1\} \), and \( T = \{j \mid v_3j \in E(H), j = 3, \ldots, k - 1\} \). So \( S \cup T \geq k \). Also we have degP_3(v3) ≥ newc(v3), (3.3) since otherwise there is \( y \in V(H) \setminus V(P_3) \) such that \( c(y, v_3) \in \text{NEWc}(v_3) \). Hence the colors of edges between H and H0 are not from c(H) nor from c(H0). Since each component is Hamiltonian and of order of \( k + 1/2 \), to avoid a rainbow C_k, by the same type of argument as in Claim 1, we must have that \( |c(H, H_0)| = 1 \)

Claim 3 For any two components H and H0 of G, \(|c(H, H_0)| = 1 \)

If there is an edge e between H and H0, incident to, say, some \( v \in V \) of color from \( \text{NEWc}(v) \), then we can make H and H0 connected by adding the edge e and deleting some edge incident to v of the same color as e in H, so the resulting graph \( G^* \) has a connected component of order \( ≥ 2(k+1/2) \), which contradicts that every connected component is of order \( ≤ k - 1 \). Hence the colors of edges between H and H0 are not from c(H) nor from c(H0). Since each component is Hamiltonian and of order of \( k + 1/2 \), to avoid a rainbow C_k, by the same type of argument as in Claim 1, we must have that \( |c(H, H_0)| = 1 \)

References