



Note on solving the transcendental equation $\sqrt{(2z-4)} = \sqrt{(x+\sqrt{C}y)} \mp \sqrt{(x-\sqrt{C}y)}$ with three unknowns for two particular values of C using continued fractions

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Abstract

In this paper, we propose the solution in \mathbb{N}^3 of the equation $\sqrt{2z-4} = \sqrt{x+\sqrt{C}y} \mp \sqrt{x-\sqrt{C}y}$ using continued fractions for two particular values of C. Our work extends the study that is already recently done on the same context. Our method will be based over the theory of continued fractions. We give the general expression of solutions by the recurrence relation.

Keywords: Transcendental equation, Continued fraction, Recurrence relation

1. Introduction

Solutions of the Pell equation play a very important role in the field of scientific research. In particular, the field of cryptology, these solutions allow to have a fairly reassured cryptographic system [8]. On the research in mathematics, its solutions also allow to find the solutions of difficult equations [9]. Recently, a new way of thinking has been implemented to solve the transcendental equation (E) : $\sqrt{2z-4} = \sqrt{x+\sqrt{C}y} \mp \sqrt{x-\sqrt{C}y}$, thanks to the solutions of the Pell equation using the continued fraction theory. Its solution which is presented in article [1] concerns only the values of $C \in \{m^2 + 1, m^2 - 1, m^2 + 2, m^2 - 2, m^2 + m \text{ et } m^2 - m\}$. The question sequent : is it not possible to express the solutions of this equation for other values of C ? This is what we will detail in this article. We will start our work by presenting the theory of solving the Pell equation, using the concept of continued fractions. Then, we will transform (E) in order to make the Pell equation $x^2 - Dy^2 = a$ (Where D is an integer which is not a perfect square) appear. Finally, we propose the solutions of (E) for two particular values of C which are not included in paper [1].

2. Preliminaries

In this part, we remind the theory on the resolution of the Pell equation, using the concept of continued fractions.

Let d be a positive integer which is not a perfect square. Let us notice by $\sqrt{d} = [a_0, \overline{a_1, a_2, \dots, a_n, 2a_0}]$, the continued fraction expansion of \sqrt{d} which is periodic. Let ℓ be the length of this period. The k^{eme} convergent of \sqrt{d} for $k \geq 0$ is given by: $r_k = \frac{p_k}{q_k} =$

$[a_0, a_1, \dots, a_k]$.

With, $\alpha_0 = \sqrt{d}$, $\alpha_k = E(\alpha_k)$ and $\alpha_{k+1} = \frac{1}{\alpha_k - \alpha_k}$, $k = 0, 1, 2, \dots$

Let (x_1, y_1) , is the fundamental solution of $x^2 - dy^2 = 1$.

Theorem 2.1

The fundamental solution of $x^2 - dy^2 = 1$ is : $(x_1, y_1) = \begin{cases} (p_{\ell-1}, q_{\ell-1}), & \text{if } m \text{ is even;} \\ (p_{2\ell-1}, q_{2\ell-1}), & \text{if } m \text{ is odd.} \end{cases}$

Theorem 2.2

Let (x_1, y_1) , is the fundamental solution of $x^2 - dy^2 = 1$. Then all positive integer solutions of the equation $x^2 - dy^2 = 1$ are given by $x_n + y_n\sqrt{d} = (x_1 + y_1\sqrt{d})^n$, with $n \geq 1$.

Theorem 2.3

Let $d \equiv 1[4]$ or $d \equiv 2[4]$ or $d \equiv 3[4]$. If (x_1, y_1) , is the fundamental solution of $x^2 - dy^2 = 1$, then the fundamental solution of $x^2 - dy^2 = 4$ is $(2x_1, 2y_1)$.

Let us then put $(u_1, v_1) = (2x_1, 2y_1)$, the fundamental solution of $x^2 - dy^2 = 4$.

Theorem 2.4.

Let $u_1 + v_1\sqrt{d}$, is the fundamental solution of $x^2 - dy^2 = 4$. Then all positive integer solutions of the $x^2 - dy^2 = 4$ is : $u_n + v_n\sqrt{d} = \frac{(u_1 + v_1\sqrt{d})^n}{2^{n-1}}$, $n \in \mathbb{N}^*$.

Corollary 2.1.

If $u_1 + v_1\sqrt{d}$ is the fundamental solution of $x^2 - dy^2 = 4$, then:

$$u_n = \frac{u_1 u_{n-1} + D v_1 v_{n-1}}{2} \text{ et } v_n = \frac{v_1 u_{n-1} + u_1 v_{n-1}}{2}$$

The complete theory on solving Pell's equation can be found in [2-7].

Now, we are going to transform the equation $\sqrt{2z-4} = \sqrt{x + \sqrt{C}y} \mp \sqrt{x - \sqrt{C}y}$ in order to show the Pell equation.

3. Transforming initial equation into three unknowns system of equations

We remember that the main objective of this article consist in extending the solution of the equation (E) which is already done in article [1].

$$\text{We have: } \sqrt{2z-4} = \sqrt{x + \sqrt{C}y} \mp \sqrt{x - \sqrt{C}y}.$$

After squaring each side of this equation, we get: $2z - 4 = 2x \pm 2\sqrt{x^2 - Cy^2}$.

$$\text{Let : } z = x + 2 \pm \sqrt{x^2 - Cy^2}.$$

$$\text{Let us then put: } x^2 - Cy^2 = \alpha^2.$$

$$\text{So : } z = x + 2 \pm \alpha^2.$$

We then obtain the following system of equations with three unknowns : (S): $\begin{cases} x^2 - Cy^2 = \alpha^2 \\ z = x + 2 \pm \alpha^2 \end{cases}$

In order to be able to consider that the first equation is of Pell, it is necessary that C is in integer which is not a perfect square, and α is an integer. Under these two conditions, the set of solutions of (S) gives the set of solutions of (E) for given value of C. In the paper [1] deals only with case $\alpha = 1$. In this paper, we will extend our study for $\alpha \in \{1,2\}$ and by successively setting the value of C by : $25m^2 + 52m + 27$ et $9m^2 + 19m + 10$, m is a natural number. It is easy to show that these two integers are perfect squares. Our main results can be stated as the following theorems.

4. Results**4.1. Solving système (S) for $\alpha = 1$**

For $\alpha = 1$, this system is defined by (S): $\begin{cases} x^2 - Cy^2 = 1 & (1) \\ z = x + 2 \pm 1 & (2) \end{cases}$

(i) For $C = 25m^2 + 52m + 27$

Theorem 4.1

Let m be a positive integer and let $C = 25m^2 + 52m + 27$.

1) The continued fraction expansion of \sqrt{C} is $\sqrt{C} = [5m + 5; \overline{5; 10m + 10}]$, $m \in \mathbb{N}$,

- 2) For all $m \in \mathbb{N}$, the fundamental solution of (1) is $(x_1, y_1) = (25m + 26, 5)$,
 3) Let us consider $\{(x_n, y_n)\}$, where : $\frac{x_n}{y_n} = \left[5m + 5; \frac{(5; 10m + 10)_{n-1}}{5}; 5\right]$ for $n \geq 2$.

Then (x_n, y_n) is the solution of $x^2 - (25m^2 + 52m + 27)y^2 = 1$

- 4) The consecutive solutions (x_n, y_n, z_n) and $(x_{n+1}, y_{n+1}, z_{n+1})$ satisfy the following system :

$$\begin{cases} x_{n+1} = (25m + 26)x_n + (125m^2 + 260m + 135)y_n \\ y_{n+1} = 5x_n + (25m + 26)y_n \\ z_{n+1} = (25m + 26)x_n + (125m^2 + 260m + 135)y_n + 2 \pm 1. \end{cases} \quad \text{for } n \geq 1.$$

- 5) The solutions (x_n, y_n, z_n) satisfy the following recurrence relation:

$$\begin{cases} x_n = (50m + 51)(x_{n-1} + x_{n-2}) - x_{n-3} \\ y_n = (50m + 51)(y_{n-1} + y_{n-2}) - y_{n-3} \\ z_n = (50m + 51)[z_{n-1} + z_{n-2} - 2 \times (2 \pm 1)] - [z_{n-3} - 2 \times (2 \pm 1)]. \end{cases}$$

Proof

Let us show (1)

For $m \geq 0$, we get : $C = 25m^2 + 52m + 27 = (5m + 5)^2 + 2m + 2$.

So, $\alpha_0 = \sqrt{C} = \sqrt{(5m + 5)^2 + 2m + 2}$.

And so, $a_0 = 5m + 5$.

And $\alpha_0 = (5m + 5) + \sqrt{C} - (5m + 5)$

$$\begin{aligned} &= (5m + 5) + \frac{(\sqrt{C} - (5m + 5))(\sqrt{C} + (5m + 5))}{(\sqrt{C} + (5m + 5))} \\ &= (5m + 5) + \frac{2m + 2}{\sqrt{C} + (5m + 5)} \\ &= (5m + 5) + \frac{1}{\frac{\sqrt{C} + 5m + 5}{2m + 2}} \end{aligned}$$

Let us put : $\alpha_1 = \frac{\sqrt{C} + 5m + 5}{2m + 2} = \sqrt{\left(\frac{5m + 5}{2m + 2}\right)^2 + \frac{1}{2m + 2}} + \frac{5m + 5}{2m + 2}$.

$$\text{For } m \geq 0, \text{ we have : } \begin{cases} \frac{5}{2} \leq \frac{5m + 5}{2m + 2} \leq \frac{5}{2} \Leftrightarrow \frac{25}{4} \leq \left(\frac{5m + 5}{2m + 2}\right)^2 \leq \frac{25}{4} \\ 0 < \frac{1}{2m + 2} \leq \frac{1}{2} \end{cases}$$

We deduce that : $5 < \alpha_1 < 5,02$.

So, $a_1 = E(\alpha_1) = 5$.

We have, $\alpha_1 = 5 + \frac{\sqrt{C} + 5m + 5}{2m + 2} - 5$

$$\begin{aligned} &= 5 + \frac{(\sqrt{C} - (5m + 5))(\sqrt{C} + (5m + 5))}{(2m + 2)(\sqrt{C} + (5m + 5))} \\ &= 5 + \frac{1}{\sqrt{C} + (5m + 5)} \end{aligned}$$

Let us put : $\alpha_2 = \sqrt{C} + (5m + 5)$.

So, $a_2 = E(\alpha_2) = 5m + 5 + 5m + 5$

$= 10m + 10$

Then, $\alpha_2 = 10m + 10 + \sqrt{C} + 5m + 5 - (10m + 10)$

$$= 10m + 10 + \frac{(\sqrt{C} - (5m + 5))(\sqrt{C} + 5m + 5)}{(\sqrt{C} + 5m + 5)}$$

$$= 5 + \frac{1}{\frac{\sqrt{C+5m+5}}{2m+2}}$$

Let us put : $\alpha_3 = \frac{\sqrt{C+5m+5}}{2m+2}$.

As $\alpha_3 = \alpha_2 = \frac{\sqrt{C+5m+5}}{2m+2}$, then the algorithm stops and the continued fraction expansion of \sqrt{C} is $\sqrt{C} = [a_0; \overline{a_1, a_2}]$.

Hence, for any natural number m : $\sqrt{25m^2 + 52m + 27} = [5m + 5; \overline{5, 10m + 10}]$.

Let us show (2)

According 1, the continued fraction expansion of $\sqrt{25m^2 + 52m + 27} = [5m + 5; \overline{5, 10m + 10}]$.

According to theorem 2.1, the fundamental solution of $x^2 - Cy^2 = 1$ is $(p_1; q_1)$.

We get : $r_1 = \frac{p_1}{q_1} = [5m + 5; 5]$

$$= 5m + 5 + \frac{1}{5}$$

$$= \frac{25m+26}{5}$$

Let: $p_1 = 25m + 26$ et $q_1 = 5$.

Hence, the fundamental solution of $x^2 - (25m^2 + 52m + 27)y^2 = 1$ is $(x_1; y_1) = (25m + 26; 5)$.

Let us show (3) and (4)

We assume that for any natural number $n \geq 2$, (x_n, y_n) is solution of

$x^2 - (25m^2 + 52m + 27)y^2 = 1$, with $\frac{x_n}{y_n} = [5m + 5; \overline{(5; 10m + 10)_{n-1}; 5}]$. We will prove that (x_{n+1}, y_{n+1}) is also solution.

We get, $\frac{x_{n+1}}{y_{n+1}} = [5m + 5; \overline{(5; 10m + 10)_n}; 5]$

$$= [5m + 5; 5; 10m + 10; \overline{(5; 10m + 10)_{n-1}}; 5]$$

$$= [5m + 5; 5; 5m + 5 + 5m + 5; \overline{(5; 10m + 10)_{n-1}}; 5]$$

$$\left[5m + 5; 5; 5m + 5 + \frac{x_n}{y_n} \right]$$

$$= 5m + 5 + \frac{1}{5 + \frac{1}{5m + 5 + \frac{x_n}{y_n}}}$$

$$= 5m + 5 + \frac{1}{\frac{5x_n + (25m+26)y_n}{x_n + 5(m+1)y_n}}$$

$$= \frac{(25m+26)x_n + (125m^2 + 260m + 135)y_n}{5x_n + (25m+26)y_n}$$

So, $\frac{x_{n+1}}{y_{n+1}} = \frac{(25m+26)x_n + (125m^2 + 260m + 135)y_n}{5x_n + (25m+26)y_n}$, $n \geq 2$.

And so : $x_{n+1} = (25m + 26)x_n + (125m^2 + 260m + 135)y_n$ and $y_{n+1} = 5x_n + (25m + 26)y_n$.

Then

$$x_{n+1}^2 - Cy_{n+1}^2 = ((25m + 26)x_n + (125m^2 + 260m + 135)y_n)^2 - C(5x_n + (25m + 26)y_n)^2$$

Let us put: $a = ((25m + 26)x_n + (125m^2 + 260m + 135)y_n)^2$
 $b = C(5x_n + (25m + 26)y_n)^2$

Then

$$a = (625m^2 + 1300m + 676)x_n^2 + (6250m^3 + 19500m^2 + 20270m + 7020)x_n y_n + (15625m^4 + 65000m^3 + 101350m^2 + 70200m + 18225)y_n^2$$

$$b = (625m^2 + 1300m + 676)x_n^2 + (6250m^3 + 19500m^2 + 20270m + 7020)x_n y_n + (15625m^4 + 101350m^2 + 70200m + 18225)y_n^2$$

$$\text{So : } x_{n+1}^2 - C y_{n+1}^2 = a - b$$

$$= x_n^2 + 0 \times x_n y_n + (-25m^2 - 52m - 27)y_n^2$$

$$= x_n^2 - (25m^2 + 52m + 27)y_n^2$$

$$= 1$$

So (x_{n+1}, y_{n+1}) is solution of $x_n^2 - (25m^2 + 52m + 27)y_n^2 = 1$.

According to equation 2, we can write $z_{n+1} = x_{n+1} + 2 \pm 1$.

$$\text{Let, } z_{n+1} = (25m + 26)x_n + (125m^2 + 260m + 135)y_n + 2 \pm 1.$$

Thus $(x_{n+1}, y_{n+1}, z_{n+1})$ is solution of $\sqrt{2z-4} = \sqrt{x + \sqrt{C}y} \mp \sqrt{x - \sqrt{C}y}$.

Let us show (5)

We assume that for $n \geq 4$, $x_n = (50m + 51)(x_{n-1} + x_{n-2}) - x_{n-3}$.

Let us show that $x_{n+1} = (50m + 51)(x_n + x_{n-1}) - x_{n-2}$.

For $n \geq 2$: $x_{n+1} = (25m + 26)x_n + (125m^2 + 260m + 135)y_n$

Let $x_{n+1} = (25m + 26)x_n + (25m^2 + 52m + 27) \times 5 \times y_n$

Or again : $x_{n+1} = x_1 x_n + D y_1 y_n$

We have : $x_{n+1} = x_1 [(50m + 51)(x_{n-1} + x_{n-2}) - x_{n-3}] + D y_1 [(50m + 51)(y_{n-1} + y_{n-2}) - y_{n-3}]$

$$= (50m + 51)[x_1(x_{n-1} + x_{n-2}) + D y_1(y_{n-1} + y_{n-2})] - x_1 x_{n-3} - D y_1 y_{n-3}$$

$$= (50m + 51)[x_1 x_{n-1} + D y_1 y_{n-1} + x_1 x_{n-2} + D y_1 y_{n-2}] - (x_1 x_{n-3} + D y_1 y_{n-3})$$

$$= (50m + 51)(x_n + x_{n-1}) - x_{n-2}$$

For all $n \geq 4$, $x_{n+1} = (50m + 51)(x_n + x_{n-1}) - x_{n-2}$.

Thus, $x_n = (50m + 51)(x_{n-1} + x_{n-2}) - x_{n-3}$.

We use the same approach to show that for $n \geq 2$, $y_n = (50m + 51)(y_{n-1} + y_{n-2}) - y_{n-3}$.

Finally, let us show that for $n \geq 2$, we get

$$z_n = (51m + 51)[z_{n-1} + z_{n-2} - 2 \times (2 \pm 1)] - [z_{n-3} - 2 \times (2 \pm 1)].$$

We know that for $n \geq 4$, one has $z_{n+1} = (25m + 26)x_n + (125m^2 + 260m + 135)y_n + 2 \pm 1$.

Let $z_n = (25m + 26)x_{n-1} + (25m^2 + 52m + 27) \times 5 \times y_{n-1} + 2 \pm 1$

Or again : $z_n = x_1 x_{n-1} + C y_1 y_{n-1} + 2 \pm 1$

We have : $z_n = x_1 [(50m + 51)(x_{n-2} + x_{n-3}) - x_{n-4}] + C y_1 [25(m + 1)(y_{n-2} + y_{n-3}) - y_{n-4}] + 2 \pm 1$

$$= (50m + 51)[x_1(x_{n-2} + x_{n-3}) + D y_1(y_{n-2} + y_{n-3})] - x_1 x_{n-4} - D y_1 y_{n-4} + 2 \pm 1$$

$$= (50m + 51)[x_1 x_{n-2} + D y_1 y_{n-2} + 2 \pm 1 + x_1 x_{n-3} + D y_1 y_{n-3} + 2 \pm 1 - 2 \times (2 \pm 1)]$$

$$\begin{aligned}
 & -(x_1 x_{n-4} + D y_1 y_{n-4} + 2 \pm 1) + 2 \times (2 \pm 1) \\
 & = (50m + 51)(z_{n-1} + z_{n-2}) - z_{n-2} + 2 \times (2 \pm 1).
 \end{aligned}$$

For $n \geq 4$, $z_n = (50m + 51)(z_{n-1} + z_{n-2}) - z_{n-2} + 2 \times (2 \pm 1)$.

(ii) For $C = 9m^2 + 19m + 10$.

In an analogous way to the previous results, we show the following theorem.

Theorem 4.2

Let m , be a positive integer and let $C = 9m^2 + 19m + 10$.

- 1) The continued fraction expansion of \sqrt{C} is $\sqrt{C} = [3m + 3; \overline{6; 6m + 6}]$
- 2) For all $m \in \mathbb{N}$, the fundamental solution of (1) is $(x_1, y_1) = (18m + 19, 6)$.
- 3) Set $\{(x_n, y_n)\}$ where: $\frac{x_n}{y_n} = [3m + 3; \overline{6; 6m + 6}]_{n-1}; 6]$,

Then (x_n, y_n) is the solution of $x^2 - (9m^2 + 19m + 10)y^2 = 1$

- 4) The consecutive solutions (x_n, y_n, z_n) and $(x_{n+1}, y_{n+1}, z_{n+1})$ satisfy the following system :

$$\begin{cases}
 x_{n+1} = (18m + 19)x_n + (54m^2 + 114m + 60)y_n \\
 y_{n+1} = 6x_n + (18m + 19)y_n \\
 z_{n+1} = (18m + 19)x_n + (54m^2 + 114m + 60)y_n + 2 \pm 1
 \end{cases}$$

- 5) The solutions (x_n, y_n, z_n) satisfy the following recurrence relation

$$\begin{cases}
 x_n = (36m + 37)(x_{n-1} + x_{n-2}) - x_{n-3} \cdot \\
 y_n = (36m + 37)(y_{n-1} + y_{n-2}) - y_{n-3} \\
 z_n = (36m + 37)[z_{n-1} + z_{n-2} - 2 \times (2 \pm 1)] - [z_{n-3} - 2 \times (2 \pm 1)].
 \end{cases}$$

Now we will solve this system for $\alpha = 2$.

4.2. Solving système (S) for $\alpha = 2$

For $\alpha = 2$, this system is defined by : $\begin{cases} x^2 - Cy^2 = 4 & (1) \\ z = x + 2 \pm 4 & (2) \end{cases}$

(i) For $C = 25m^2 + 52m + 27$.

Lemma 4.1

Let $C = 25m^2 + 52m + 27$. If m is even, then $C \equiv 3[4]$

Theorem 4.3

If m is even, then the fondamentat solution of $x^2 - Cy^2 = 4$ is $(u_1, v_1) = (50m + 52, 10)$.

Proof

Proof of lemma 4.1

We get, $C = 25m^2 + 52m + 27 \equiv m^2 + 3[4]$.

If m is even, that is to say $m = 2k$, with $k \in \mathbb{N}$, then $m^2 = 4k^2$.

So $4|m^2$, thus, $C \equiv 3[4]$.

Proof of theorem 4.3

According to lemma 4.1, if m is even, then, $C \equiv 3[4]$. According to the theorem 2.3, and the second point of theorem 4.1, $(u_1, v_1) = (50m + 52, 10)$ is the fondamental of $x^2 - (25m^2 + 52m + 27)y^2 = 4$.

Theorem 4.4

Let m be a positive integer and let $C = 25m^2 + 52m + 27$.

- 1) The consecutive solutions (x_n, y_n, z_n) and $(x_{n+1}, y_{n+1}, z_{n+1})$ satisfy the following system

$$\begin{cases} u_{n+1} = (25m + 26)u_n + (125m^2 + 260m + 135)v_n \\ v_{n+1} = 5u_n + (25m + 26)v_n \\ z_{n+1} = (25m + 26)u_n + (125m^2 + 260m + 135)v_n + 2 \pm 4 \end{cases}$$

2) The solutions (x_n, y_n, z_n) satisfy the following recurrence relation

$$\begin{cases} u_n = (50m + 51)(u_{n-1} + u_{n-2}) - u_{n-3} \\ v_n = (5m + 51)(v_{n-1} + v_{n-2}) - v_{n-3} \\ z_n = (50m + 51)[z_{n-1} + z_{n-2} - 2 \times (2 \pm 4)] - [z_{n-3} - 2 \times (2 \pm 4)] \end{cases}$$

Proof

According to the corollary 2.1, we get:

$$u_n = \frac{u_1 u_{n-1} + D v_1 v_{n-1}}{2} \text{ et } v_n = \frac{v_1 u_{n-1} + u_1 v_{n-1}}{2}$$

$$\text{So: } \begin{cases} u_{n+1} = \frac{u_1 u_n + D v_1 v_n}{2} \\ v_{n+1} = \frac{v_1 u_n + u_1 v_n}{2} \end{cases}$$

$$\text{Let: } \begin{cases} u_{n+1} = \frac{(50m+52)u_n + (25m^2+52m+27) \times 10 \times v_n}{2} \\ v_{n+1} = \frac{10u_n + (50m+52)v_n}{2} \end{cases}$$

From where, for $n \geq 1$, we have: $\begin{cases} u_{n+1} = (25m + 26)u_n + (125m^2 + 260m + 135)v_n \\ v_{n+1} = 5u_n + (25m + 26)v_n \end{cases}$

For the third equation: $z_{n+1} = u_{n+1} + 2 \pm 4$.

So for $n \geq 1$, $z_{n+1} = (25m + 26)u_n + (125m^2 + 260m + 135)v_n + 2 \pm 4$.

For the second point, we prove in a way analogous to the fifth point of theorem 4.1.

(ii) For $C = 9m^2 + 19m + 10$

Lemma 4.2

Let $C = 9m^2 + 19m + 10$. If $m = 4k$ or $m = 4k + 1$, with $k \in \mathbb{N}$, then $C \equiv 2[4]$.

Theorem 4.5

If $m = 4k$ or $m = 4k + 1$, then, the fundamental solution of $x^2 - Cy^2 = 4$ is $(u_1, v_1) = (36m + 38, 12)$

Theorem 4.6

Let m be a positive integer and let $C = 9m^2 + 19m + 10$.

1) The consecutive solutions (x_n, y_n, z_n) and $(x_{n+1}, y_{n+1}, z_{n+1})$ satisfy the following system

$$\begin{cases} u_{n+1} = (18m + 19)u_n + (54m^2 + 114m + 60)v_n \\ v_{n+1} = 6u_n + (18m + 19)v_n \\ z_{n+1} = (18m + 19)u_n + (54m^2 + 114m + 60)v_n + 2 \pm 4 \end{cases}$$

2) The solutions (x_n, y_n, z_n) satisfy the following recurrence relation

$$\begin{cases} x_n = (36m + 37)(x_{n-1} + x_{n-2}) - x_{n-3} \\ y_n = (36m + 37)(y_{n-1} + y_{n-2}) - y_{n-3} \\ z_n = (36m + 37)[z_{n-1} + z_{n-2} - 2 \times (2 \pm 4)] - [z_{n-3} - 2 \times (2 \pm 4)] \end{cases}$$

5. Conclusion

In this paper, we presented the resolution of the transcendental equation $\sqrt{x + \sqrt{C}y} \mp \sqrt{x - \sqrt{C}y}$ thanks for the solutions of the Pell equation, using of concept the continued fraction for two particular values of C . Our work is an extension of that which is already done in the literature. In future work, we will consider studying this same equation in the general case.

Références

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