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Application of the Laplace and Fourier Transformation in the resolution of PDE

Yosra Annabi
Free Researcher, Street Cheikh Mohamed Hedi Belhadej, Romana, Tunisia, yosra.annabi@gmail.com

* Corresponding Author: **Yosra Annabi**

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Abstract

The solution of partial differential equations has been studied by several scientists since the nineteenth century. Several intuitive attempts have been made using constructive methods such as the variable separation method. A more general study was carried out by George Green in 1828 for the needs of electromagnetism. He was able to develop a general study of elementary solution or fundamental solution; of a linear differential equation with constant coefficients, or of a linear partial differential equation with constant coefficients. This made it possible to show that the solutions have a particular integral writing and form an affine space. Thus the general solution of a PDE is the sum of a particular solution and the solution of the homogeneous equation, i.e. the second member is zero. This article presents demonstrations of such results.

Keywords: Fourier transform, Laplace transform, EDO, PDE, distribution, convolution

1.Mathematical Tools

1.1. The distributions

A distribution, also called a generalized function, is an object that generalizes the notion of function and measure.

Definition 1: If U is an open of $\mathfrak{R}^n$, we call distribution in the open set U a linear form u, defined on the function space $C_c^\infty(U)$, verifying the following continuity condition : for any compact K of U, there exists $p \in N$ and $C > 0$ such that, for any $g \in C_c^\infty(U)$ with support in K.

$$|u(g)| \leq C \sup_{x\in K,|\alpha|\leq p} |\partial^\alpha g(x)| \tag{1}$$

$$with\ \partial^\alpha g = \frac{\partial|\alpha|g}{\partial x_1^{\alpha_1}\dots\partial x_n^{\alpha_n}}.$$

Example 1: 1. Locally integrable functions are distributions.
2. Dirac masses are distributions
We denote δ a Dirac distribution. It verifies the fundamental property that, for any function $x \rightarrow \phi(x)$ smooth:

$$\langle\delta,\varphi\rangle = \varphi(0) \tag{2}$$

1.2. Approximation of the unit for the convolution

To overcome the absence of a neutral element for the convolution on $L^1(\lambda_d)$, approximate units are introduced, that is to say sequences of functions $\left(\alpha_n\right)_{n\geq 1}$ behaving asymptotically as a unit. In other words, it is desired that

$$\alpha_n * f \quad \approx \quad f \text{ , when } n \to +\infty \tag{3}$$

Definition 2 *A sequence* $(\alpha_n)_{n\geq 1}$ *of an element of* $L^1(\lambda_d)$ *is an approximation of unity if it verifies*

1. *for all* $n \geq 1$, $\int_{\mathbb{R}^d} \alpha_n d\lambda_d = 1$.
2. $\sup\limits_{n\geq 1} \int_{\mathbb{R}^d} |\alpha_n| d\lambda_d < +\infty$
3. *for all* $\varepsilon > 0$, $\lim\limits_{n} \int_{\{|x|\geq\varepsilon\}} |\alpha_n| d\lambda_d = 0$

Apart from the sequences, we also consider families $(\alpha_t)_{t\in\mathfrak{R}_+}$ of approximation of the unit indexed by $\mathfrak{R}_+^*$. In this case, the third condition becomes *t* tends to 0 instead of *n* to $+\infty$.

Example 2: The Laplace kernel for the unity approximation

We denote by |.| the canonical Euclidean norm on $\mathfrak{R}^d$. The Laplace kernel is set to $\mathfrak{R}$ by

$$\forall t > 0, \alpha_t(x) \quad := \quad \frac{1}{2t} e^{-\frac{|x|}{t}} \tag{4}$$

$$from\ \alpha_1(x) = \tfrac{1}{2} e^{-|x|}$$

1.3 The functions of Green

A Green function is a solution of a linear differential equation with constant coefficients, or of a linear partial differential equation with constant coefficients, with a second member equal to the Dirac mass. These Green functions are most often distributions. A Green function is also called an elementary solution or a fundamental solution:

Definition 3: We call Green's function any solution G of the linear partial differential equation:

$$\mathfrak{D}(g) = \delta \tag{5}$$

It will be noted however that the set of solutions of the equation 5, that is to say

$$\{G, \text{ checking } \mathfrak{D}\, G \;=\; \delta\} \tag{6}$$

Is an affine space. Thus the general solution of the equation 5 is the sum of a particular solution and the general solution of

$$\mathfrak{D}\, \varphi \quad = \quad 0 \tag{7}$$

The uniqueness of the solution of 5 i.e. of a Green function depends on:

1. The linear partial differential equation
2. The boundary conditions
3. The initial condition for the evolution PDEs

Theorem 3: If the Green function G: x ›⟶ G(x) is known, then the solution ϕ: x ›⟶ ϕ(x) of the equation

$$\mathfrak{D}\, \varphi(x) \quad = \quad j(x) \tag{8}$$

Is written in the form of a convolution product:

$$\begin{aligned} \varphi(x) \quad &= \quad (G * j)(x) \\ &= \quad \int G(x-y) j(y)\, dy \end{aligned} \tag{9}$$

Demonstration 4: (Theorem 3) As *D* acts on the variable x, we obtain

$$\mathfrak{D}\, \varphi(x) \quad = \quad \int [\mathfrak{D}\, G(x-y)]\ j(y)\, \mathrm{d}y \tag{10}$$

However, the Green function G verifies the equation 5, so

$$\begin{aligned} \mathfrak{D}\,\varphi(x) &= \int \delta(x-y)\, j(y)\,\mathrm{d}y \\ &= j(x) \end{aligned} \tag{11}$$

1.4. Kernel of an integral operator

An integral operator or kernel operator is a linear operator defined using a parametric integral over certain functional spaces. The image of a function by such an operator is therefore another function, the domain of which can be very different.

Definition 4: The general form of an integral operator is given by the following expression:

$$S(f)(t) = \int_A f(x) K(x,t)\,\mathrm{d}\mu(x) \tag{12}$$

In which the function K is called the kernel of the operator.
In many common examples, the integration domain A is a real interval and the associated measure is that of Lebesgue.

Theorem 5: Let L_K be the linear operator on $L_2^{\nu}(\chi)$ defined by:

$$\forall f \in L_2^{\nu}(\chi) \quad , \quad (L_K f)(x) = \int K(x,t) f(t)\, d\nu(t) \tag{13}$$

Let ($\lambda 1, \lambda 2,...$) be the decreasing eigenvalues of L_K, and ($\psi 1, \psi 2,...$) the corresponding eigenvectors. So, we have for all x,y:

$$\begin{aligned} K(x,y) &= \sum_{k=1}^{\infty} \lambda_k \psi_k(x) \psi_k(y) \\ &= \langle \phi(x), \phi(y) \rangle_{l2} \end{aligned} \tag{14}$$

$$\textit{with } \phi \textit{ defined by } \phi(x) = \left(\sqrt{\lambda_k}\,\psi_k(x)\right)_{k \in \mathbb{N}^*}.$$

1.4.1. The Laplace transformation

We call causal function any function defined on $\mathfrak{R}$, null on $]-\infty,0[$ and continues piecewise on $[0,+\infty[$. The step-unit function is the causal function U defined by U(t) = 0 if t < 0 and U(t) = 1 if t ≥ 0. In particular if f: $\mathfrak{R} \to \mathfrak{R}$ is any function, the function $f \times U$ is a causal function.

Definition 5: If f is a causal function, the Laplace transform of f is defined by

$$\begin{aligned} \mathfrak{L}(f)(p) &= \int_0^{+\infty} e^{-pt} f(t)\, dt \\ &= \lim_{x \to +\infty} \int_0^x e^{-pt} f(t)\, dt \end{aligned} \tag{15}$$

For the values of p for which this integral converges.

Example 6: According to the definition 4, the kernel of the Laplace transform

$$K(x,y) = e^{-xy} \tag{16}$$

Proposition 1: If f is a causal function that checks, for any x ≥ 0,

$$|f(x)| \leq Me^{ax} \tag{17}$$

Then the Laplace transform $L(f)(p)$ exists for any p > a.
Proposition 2: If f is a locally integrable function, defined on $\mathfrak{R}_+$, with values in C, then its Laplace transform is the function

$$\mathfrak{L}(f)(z) = \int_0^{+\infty} f(t) e^{-zt}\, dt \tag{18}$$

$$\textit{with } z = x + iy.$$

Application 7: 1. The Laplace transform of the derivative is:

$$\mathfrak{L}\left(\frac{\partial}{\partial t}u(x,.)\right)(z) \quad = \quad z\mathfrak{L}(u)(x,t)-u(x,0) \tag{19}$$

2. The Laplace transform of the second derivative is:

$$\mathfrak{L}\left(\frac{\partial^2}{\partial t^2}u(x,.)\right)(z) \quad = \quad z^2\mathfrak{L}(u)(x,t)-zu(x,0)-\frac{\partial\ u}{\partial t}(x,0) \tag{20}$$

1.4.2. The Fourier transformation

Definition 6: Fourier series

Given a periodic function f: $\Re\rightarrow C$ of period 2π and bounded. We call Fourier series of f the formal series of the form:

$$f(t) \quad = \quad a_0+\sum_{n=1}^{\infty}(a_n cos(nwt)+b_n sin(nwt)) \tag{21}$$

The coefficients a_0, an and bn are independent of time and are given by the following integrals:

$$a_0 \quad = \quad \frac{1}{T}\int_0^T f(t)\,dt \tag{22}$$

$$a_n \quad = \quad \frac{2}{T}\int_0^T f(t)cos(nwt)\,dt \tag{23}$$

$$b_n \quad = \quad \frac{2}{T}\int_0^T f(t)sin(nwt)\,dt \tag{24}$$

Remark 1: In complex notation, any physical periodic signal can be written

$$f_T(t) \quad = \quad \sum_{n=-\infty}^{\infty} c_n\, e^{in2\pi\nu t} \tag{25}$$

$$with\ \nu=\frac{1}{T}\ and$$

$$c_n \quad = \quad \frac{1}{T}\int_{-T/2}^{T/2} f(t)\,e^{-in2\pi\nu t}\,dt \tag{26}$$

$$with\ n\in\mathbb{Z}.$$

Theorem 8: Any physical signal can be decomposed into a Fourier integral, of the form

$$f(t) \quad = \quad \int_{-\infty}^{+\infty}\hat{f}(\nu)\,e^{-in2\pi\nu t}\,d\nu \tag{27}$$

Where $\hat{f}(\nu)$ denotes the Fourier transform of the signal. $\hat{f}(\nu)$ is a continuous complex-valued function, defined by

$$\hat{f}(\nu) \quad = \quad \int_{-\infty}^{+\infty} f(t)\,e^{-in2\pi\nu t}\,dt \tag{28}$$

Remark 2: The theorem 8 concerns the summable square signals for which $\int_{\Re}|f(t)|^2dt$ is finite, that is to say signals which transport a finite energy as is the case in physics.

Example 9: According to the definition 4, the general writing of the kernel of the Fourier transform is

$$K(x,y) = e^{-ixy} \tag{29}$$

2 Green function (Fundamental solution) of the heat equation

The heat equation is a parabolic evolution PDE.

Theorem 10: Let I be an open such that $0 \in \bar{I}$. The following system is considered with the initial Cauchy-type condition of the form:

$$\begin{cases} \partial_t u(x,t) - \Delta u(x,t) & = & 0\,,\, x \in \mathbb{R}^d, t \in I \\ u(x,0) & = & v(x) \end{cases} \tag{30}$$

So, a solution of the heat equation is:

$$u(x,t) \quad = \quad \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} v(y)\; e^{-|x-y|^2/4t} dy\,,\; \forall x \in \mathbb{R}^d, \forall t > 0 \tag{31}$$

Demonstration 11: Suppose that 30, admits a solution and that we can apply the Fourier transform with respect to $x = (x1,..., xd) \in \mathfrak{R}^d$, for any $\xi \in \mathfrak{R}^d$:

$$\begin{aligned} \hat{u}(\xi,t) & = \mathfrak{F}(u(.,t))(\xi) \\ & = \int_{\mathbb{R}^d} u(x,t)\; e^{-ix\xi}\; dx \end{aligned} \tag{32}$$

We apply the Fourier transformation to the two members of the first equality of the system 30, we obtain:

$$\mathfrak{F}(\partial_t u(.,t) - \Delta u(.,t))(\xi) \quad = \quad 0\,,\, \forall \xi \in \mathbb{R}^d, \forall t \in I \tag{33}$$

By linearity, we find

$$\partial_t \mathfrak{F}(u(.,t))(\xi) - \mathfrak{F}(\Delta u(.,t))(\xi) \quad = \quad 0\,,\, \forall \xi \in \mathbb{R}^d, \forall t \in I \tag{34}$$

As the Fourier transformation of the Laplacian operator Δ is

$$\mathfrak{F}(\Delta u(.,t))(\xi) \quad = \quad -|\xi|^2 \hat{u}(\xi,t) \tag{35}$$

The equation 34 becomes:

$$\partial_t \hat{u}(\xi,t) + |\xi|^2 \hat{u}(\xi,t) \quad = \quad 0\,,\, \forall \xi \in \mathbb{R}^d, \forall t \in I \tag{36}$$

Finally, we obtain a new system equivalent to 30:

$$\begin{cases} \partial_t \hat{u}(\xi,t) + |\xi|^2 \hat{u}(\xi,t) & = & 0\,,\, \forall \xi \in \mathbb{R}^d, \forall t \in I \\ \hat{u}(\xi,0) & = & \hat{v}(\xi) \end{cases} \tag{37}$$

This linear ordinary derivative equation of order 1 admits a solution given by:

$$\hat{u}(\xi,t) \quad = \quad \hat{v}(\xi)\; e^{-|\xi|^2 t} \tag{38}$$

Remark 3: We know that if

$$w(x) \quad = \quad e^{-|x|^2 t} \tag{39}$$

Then

$$\hat{w}(x) \quad = \quad \pi^{\frac{d}{2}} e^{-|\xi|^2/4} \tag{40}$$

Which allows us to write $e^{-|\xi|^2 t} = F(a(., t))(\xi)$, This is the Gaussian kernel also called the heat kernel:

$$G(x,t) \quad = \quad \frac{1}{(4\pi t)^{d/2}} e^{-|x|^2/4t} \tag{41}$$

By applying the inverse Fourier formula, we find the result 31.

Remark 4: The theorem 3 makes it possible to construct a solution of the equation with a second member.

Remark 5: The heat kernel is a Green function of the heat equation over a specified domain, possibly with appropriate boundary conditions. It is also one of the main tools in the study of the Laplacian spectrum.
The heat core represents the evolution of the temperature equal to one unit of heat at a point at the initial time.

Theorem 12: It is assumed that u is bounded continuous on R^d then the function

$$u(x,t) = \frac{1}{(4\pi t)^{d/2}} \int_{\mathbb{R}^d} v(y)\, e^{-|x-y|^2/4t} dy\,,\ \forall x \in \mathbb{R}^d, \forall t > 0 \tag{42}$$

is the solution of the following system:

$$\left\{ \begin{array}{lcl} \partial_t u(x,t) - \Delta u(x,t) & = & 0\,,\ x \in \mathbb{R}^d, t > 0 \\ \forall x_0 \in \mathbb{R}^d,\ \lim\limits_{(x,t)\to(x_0,0)} u(x,t) & = & v(x_0) \end{array} \right. \tag{43}$$

Demonstration 13: Let's show the initial condition. Let $x_0 \in \mathfrak{R}^d$ be fixed. We pose

$$\eta = \frac{y-x}{\sqrt{4t}} \tag{44}$$

And

$$d\eta = \frac{1}{(4t)^{\frac{d}{2}}} dy \tag{45}$$

Hence

$$u(x,t) = \frac{1}{(\pi t)^{d/2}} \int_{\mathbb{R}^d} v\left(\sqrt{4b}\eta + x\right) e^{-|x|^2} d\eta \tag{46}$$

It is recalled that

$$\frac{1}{(\pi)^{d/2}} \int_{\mathbb{R}^d} e^{-|\eta|^2} d\eta = 1 \tag{47}$$

Hence

$$u(x,t) - v(x_0) = \frac{1}{(\pi)^{d/2}} \int_{\mathbb{R}^d} \left(v(x+\sqrt{4t}\eta) - v(x_0)\right) e^{-|\eta|^2} d\eta \tag{48}$$

Let be $\varepsilon_0 > 0$. For $R > 0$, we have

$$\begin{aligned} |u(x,t) - v(x_0)| \leq & \frac{1}{(\pi)^{d/2}} \int_{|\eta|>R} 2\|v\|_\infty\, e^{-|\eta|^2} d\eta \\ + & \frac{1}{(\pi)^{d/2}} \int_{|\eta|<R} \left|v\left(x+\sqrt{4t}\eta\right) - v(x_0)\right| e^{-|\eta|^2} d\eta \end{aligned} \tag{49}$$

However, there exists R_0 such that $R \geq R_0$ we have

$$\frac{1}{(\pi)^{d/2}} \int_{|\eta|>R} 2\|v\|_\infty\, e^{-|\eta|^2} d\eta \leq c \tag{50}$$

Let's note

$$J(x,t) = \frac{1}{(\pi)^{d/2}} \int_{|\eta|<R} \left|v\left(x+\sqrt{4t}\eta\right) - v(x_0)\right| e^{-|\eta|^2} d\eta \tag{51}$$

And let's set $R = R_0$, we have

$$0 \le J(x,t) \le \sup|\eta| \le R_0 \left|v\left(x+\sqrt{4t}\eta\right) - v(x_0)\right| \tag{52}$$

Hence

$$\left|x+\sqrt{4t}\eta - x_0\right| \quad \le \quad |x-x_0| + 2R_0\sqrt{t} \tag{53}$$

We know that there exists δ > 0 such that if $\left|a - x_0\right| < \delta$ then $|v(a) - v(x_0)| \le \varepsilon$ so if

$$\begin{cases} |x-x_0| & \le \quad \frac{\delta}{2} \\ 0 < t \le \frac{\delta^2}{4R_0^2} & \end{cases} \tag{54}$$

Then

$$\left|v\left(x+\sqrt{4t}\eta\right) - v(x_0)\right| \quad \le \quad c\,, \forall|\eta| \le R_0 \tag{55}$$

Thus, the initial condition is verified.

3. Green function (Fundamental solution) of the wave equation

The wave equation is a hyperbolic evolution PDE.

Theorem 14: We consider $d\sigma_R$ the uniform measure on the sphere of center O and radius R in $\mathfrak{R}^2$. The distribution E defined by:

$$\forall \varphi \in \mathscr{S}\left(\mathbb{R}^3 \times \mathbb{R}^+\right) \quad , \quad < E, \varphi >= \int_{t=0}^{+\infty} \int_{\mathbb{R}^3} \frac{1}{4\pi t} \varphi(x,t)\ d\sigma_R(x) dt \tag{56}$$

Belongs to

$$\mathscr{S}\left(\mathbb{R}^3 \times \mathbb{R}^+\right) \tag{57}$$

And check

$$\mathfrak{D}E \quad = \quad \delta_0 \tag{58}$$

$$with\ \mathfrak{D} = \frac{\partial^2}{\partial t^2} - \frac{1}{c^2}\Delta_x.$$

Demonstration 15:

First step : Fourier transformation of the equations 58.

We denote F the Fourier transform in space and apply it to the equality 58. On the one hand, we have:

$$\mathfrak{F}(\mathfrak{D}E) \quad = \quad \partial_{tt}^2 \mathfrak{F}E + |\xi|^2\, \mathfrak{F}E \tag{59}$$

On the other hand, ∀ ϕ ∈ S,

$$\begin{aligned} < \mathfrak{F}\delta_0, \varphi > \quad &= \quad < \delta_0, \mathfrak{F}\varphi > \\ &= \quad \int e^{-ix\xi} \varphi(x,t)\ dx \Big|_{t=0,\xi=0} \\ &= \quad \int \varphi(x,0)\ dx \end{aligned} \tag{60}$$

So,

$$\mathfrak{F}\delta_0 = \delta_{t=0}\otimes\mathbb{1} \tag{61}$$

Finally, E is solution in $\mathscr{S}'(\mathbb{R}^3\times\mathbb{R}^+)$ of 58 if and only if $E^{\sim} = FE$ verifies:

$$\partial_{tt}^2\tilde{E}+|\xi|^2\tilde{E} = \delta_{t=0}\otimes\mathbb{1} \tag{62}$$

Second step

Construction of E˜.

We are looking for E˜ in the form

$$\tilde{E}(\xi,t) = H(t)\left(a(\xi)cos(t|\xi|)+b(\xi)sin(t|\xi|)\right) \tag{63}$$

With

$$H:t \mapsto \begin{cases} 1\ si\ t>0 \\ 0\ si\ t<0 \end{cases} \tag{64}$$

Then,

$$\partial_t\tilde{E}(\xi,t) = \delta_{t=0}\otimes a(\xi)+H(t)[.|\xi|a(\xi)sin(t|\xi|)+|\xi|b(\xi)cos(t|\xi|)] \tag{65}$$

And

$$\partial_{tt}^2\tilde{E}(\xi,t) = \delta'_{t=0}\otimes a(\xi)+\delta_{t=0}\otimes|\xi|b(\xi)-|\xi|^2\tilde{E} \tag{66}$$

As E˜ verifies 62, for identification, we notice that $a(\xi)\equiv 0$ and $b(\xi) = 1/|\xi|$.

Finally,

$$\tilde{E}(\xi,t) = H(t)\frac{sin(t|\xi|)}{|\xi|} \tag{67}$$

Third step

Let's show that E˜ is measurable and E˜ ∈ $\mathscr{S}'(\mathbb{R}^3\times\mathbb{R}^+)$. Let be $\phi\in\mathscr{S}(\mathbb{R}^3\times\mathbb{R}^+)$

$$<\tilde{E},\varphi> = \int_{t=0}^{+\infty}\int_{\mathbb{R}^3}\frac{sin(t|\xi|)}{|\xi|}\varphi(\xi,t)\ d\xi\ dt \tag{68}$$

Then

$$|<\tilde{E},\varphi>| \leq \left(\int_{t=0}^{+\infty}\int_{\mathbb{R}^3}\frac{t\ d\xi dt}{(1+t^2+|\xi|^2)^{n+1}}\right)\sup_{(t,\xi)}\left[\left(1+t^2+|\xi|^2\right)^{n+1}|\varphi(\xi,t)|\right] \tag{69}$$

Therefore, E˜ is measurable.

Also, by construction, E˜ checks 62 so, E = F^{-1}E˜ belongs to $\mathscr{S}'(\mathbb{R}^3\times\mathbb{R}^+)$ and checks 58.

Fourth step

Let's show that the constructed solution is written in the form 56.

The measure $d\sigma_R$ is with compact support in $\mathfrak{R}^3$. So $f(d\sigma_R)$ is a function C^∞. More precisely,

$$\mathfrak{F}\left(\frac{d\sigma_R}{4\pi R^2}\right)(\xi) = \frac{sin(R|\xi|)}{R|\xi|} \tag{70}$$

This equality is demonstrated in the proposition 3 which is at the end of this demonstration. Thus, ∀ ϕ ∈ S

$$\begin{aligned} < E,\varphi > &= < \tilde{E},\mathfrak{F}^{-1}\varphi > \\ &= \int_{t=0}^{+\infty}\int_{\mathbb{R}^3}\frac{sin(t\ |\xi|}{|\xi|}\ \mathfrak{F}^{-1}\varphi(\xi,t)\,d\xi dt \\ &= \int_{t=0}^{+\infty} < \mathfrak{F}(\frac{d\sigma_R}{4\pi t}),\mathfrak{F}^{-1}\varphi(.,t) > \ dt \\ &= \int_{t=0}^{+\infty}\frac{1}{4\pi t}\int_{|x|=t}\varphi(x,t)\ d\sigma_R(x)\,dt \end{aligned} \tag{71}$$

Which ends the demonstration.

Remark 6: In dimension 3, the solution E is to support on the cone {|x| = t}.

Proposition 3: We have

$$\mathfrak{F}\left(\frac{d\sigma_R}{4\pi R^2}\right)(\xi) = \frac{sin(R|\xi|)}{R|\xi|} \tag{72}$$

Demonstration 16: We have $d\sigma_R$ is rotation invariant, so $f(d\sigma_R)$ too. Indeed, for all A orthogonal,

$$\begin{aligned} \mathfrak{F}(d\sigma_R)(A\xi) &= < d\sigma_R, e^{-i(A\xi).x} > \\ &= < d\sigma_R, e^{-i\xi x} > \end{aligned} \tag{73}$$

On the other hand,

$$\begin{aligned} \mathfrak{F}(d\sigma_R)(\xi) &= \mathfrak{F}(d\sigma_R)(0,0,|\xi|) \\ &= \int_{|x|=R} e^{-i|\xi|x_3}\ d\sigma_R(x) \end{aligned} \tag{74}$$

We make the change of variables to the spherical reference frame by posing

$$\begin{aligned} x_1 &= R\ sin\theta\ cos\varphi \\ x_2 &= R\ sin\theta\ sin\varphi \\ x_3 &= R\ cos\theta \end{aligned} \tag{75}$$

For all 0 < θ < π and 0 < ϕ < 2π. Hence,

$$\int_{|x|=R} e^{-i|\xi|x_3}\ d\sigma_R(x) = 2\pi\int_0^{\pi} e^{-i|\xi|Rcos\theta}R^2\ sin\theta\,d\theta \tag{76}$$

We put t = −cosθ, we find

$$\begin{aligned} 2\pi\int_0^{\pi} e^{-i|\xi|Rcos\theta}R^2\ sin\theta\,d\theta &= 2\pi R^2\int_{-1}^{1} e^{i|\xi|Rt}dt \\ &= 4\pi R^2\frac{sin(|\xi|R)}{|\xi|R} \end{aligned} \tag{77}$$

Conclusion 17: A solution E of the wave equation is:

$$\forall\varphi\in\mathscr{S}\left(\mathbb{R}^3\times\mathbb{R}^+\right)\quad,\quad < E,\varphi > = \int_{t=0}^{+\infty}\int_{\mathbb{R}^3}\frac{1}{4\pi t}\varphi(x,t)\ d\sigma_t(x)dt \tag{78}$$

4. Conclusion

This article presents examples of Green's functions. These are solutions in integral form of the heat equation and the wave equation. They are also called Green's functions.

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