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Results on non expansive type mapping

Dr. Abha Tenguria [1*], **Mayuri Nema** [2]
[1] Professor, Department of Mathematics, MLB Girls College Bhopal, Madhya Pradesh, India
[2] Research Scholar, Barkatullah University Bhopal, Madhya Pradesh, India

* Corresponding Author: **Dr. Abha Tenguria**

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Abstract
In this paper author presents fixed point and coincidence point theorem for single valued mapping which satisfy the condition of non-expansive type mapping in the context of asymptotically regular mappings.

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1. Introduction

Suppose (X, d) is a metric space and $f: X \to X$ then we say that f is contraction correspondingly non-expansive mapping if there exists a non- negative real number $k \leq 1$ then the inequality $d(fx, fy) \leq k\, d(x, y)$ which satisfy for any $x, y \in X$ by Banach contraction principle X is a complete space and f has a unique fixed point in X. Non-expansive mapping has a Lipschitz constant equal to one.
In complete metric space, non-expansive mapping has more than one fixed point or may not have any fixed point.

Definition 1.1: For a function $f(x)$, a fixed point is a value of x such that $f(x) = x$.
Example: Consider the function $f(x) = sin(x)$. At $x = 0$, $f(0) = sin(0) = 0$
Since $f(0) = 0$, it satisfies the condition $f(x) = x$. Hence, 0 is a fixed point of the function.
However, not all functions have fixed points. For instance, consider $f(x) = x + 1$. The addition of 1 ensures that no value of x satisfies $f(x) = x$.
Alternatively, we can think of $f(x)$ as representing y. Fixed points are the values of x where, $x = y$.

Definition 1.2: Let (X, d) be a metric space. A mapping $T: X \to X$ is said to be an expansive mapping (or expansion) if, for all $x, y \in X$ the following condition is satisfied: $d(T(x), T(y)) \geq d(x, y)$

Definition 1.3: [5] A contraction mapping (also referred to as a contraction or contractor) on a metric space (X, d) is a function $f: X \to X$ that satisfies the following property:
There exists a real number $0 \leq k < 1$ such that, for all $x, y \in X$

$$d(f(x), f(y)) \leq k\, d(x, y)$$

The smallest value of k satisfying this condition is known as the Lipschitz constant of f Contractive mappings are sometimes referred to as Lipschitzian maps.

If the above inequality holds for $k \leq 1$ instead, the mapping is called a non-expansive map.

Definition 1.4: In a metric space $\boldsymbol{X}$ let f and g be two self-mapping then f and g are said to be compatible if the condition satisfy $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, where $\{x_n\}$ is a sequence such that

$$\lim_{n\to\infty} f x_n = \lim_{n\to\infty} gx_n = \text{t} \in \mathbf{X}.$$

Let T and f be self-mappings on a metric space $(\boldsymbol{X}, d)$ such that $g(\boldsymbol{X}) \subset f(\boldsymbol{X})$. Then, for any $x_0 \in \boldsymbol{X}$, we take a point $x_1 \in \boldsymbol{X}$ such that $fx_1 = gx_0$. Carry on with this process, suppose a sequence $\{x_k\}$ in $\boldsymbol{X}$ such that $fx_{k+1} = gx_k, k = 0, 1, 2,$ and the sequence $\{fx_n\}$ is know as $g - sequence$ with initial point x_0.

Definition 1.5: Suppose $g \; and \; f$ be self-mappings on a metric space $(\boldsymbol{X}, d)$ such that $g(\boldsymbol{X}) \subset f(\boldsymbol{X})$. Then the mapping g is said to be asymptotically $f - regular$ at point $x_0 \in \boldsymbol{X}$, if

$$\lim_{n\to\infty} \text{d}(\text{f}x_n, \text{f}x_{n+1}) = 0,$$

where $\{fx_n\}$ is a $g - sequence$ with initial point x_0.

2. Main Result

In this section, by the help of non - expansive type condition we have proved important result of fixed point theory for a single valued mapping in the context of asymptotically regular mapping.

2.1 Fixed Point Theorem For A Single Valued Mapping

In this part for all $a, b \; \epsilon \; X$. Now consider the following non – expansive type condition:

$$d(fa, fb) \leq \alpha \, d \, [(a, fa) + d(b, fb)] + \beta[(a, fa) + d(b, fb)] + \gamma \, d(a, b) + \xi[M(a, b) + hm(a, b)] \tag{1}$$

$$\gamma \geq 0, \alpha, \beta, \xi > 0, 0 < h < 1 \tag{2}$$

$$2\alpha + 2\beta + \gamma + 2\xi = 1 \tag{3}$$

$$M(a, b) = \max\{(d(a, fb), d(b, fa)\},$$

And $m(a, b) = \min\{(d(a, fb), d(b, fa)\}$

The idea of asymptotically regular mapping for a point in the hilbert space was bring in by Browder and Petryshyn. F is a mapping on a metric space (X, d) is alleged asymptotically regular at $x \; \epsilon \; X$, if

$$\lim_{n\to\infty} d(f^n x, f^{n+1} x) = 0.$$

Proposition 1: Let f be a self–mapping on X satisfying the condition (1) with (2) and (3). Then f is asymptotically regular at each point in X.

Proof: Suppose a_0 be any point in X, we define a sequence $\{a_n\}$
Such that From (1)

$$a_{n+1} = fa_n = f^n a_0, for \; n \; \epsilon \; N \cup \{0\}$$

$d(a_n, a_{n+1}) = d(fa_{n-1}, fa_n)$ $\quad d(a_n, a_{n+1}) \leq \alpha[d(a_{n-1}, a_n) + d(a_n, a_{n+1})] + \beta[d(a_{n-1}, a_{n+1}) + d(a_n, a_n)] + \gamma d(a_{n-1}, a_n) + \xi[M(a_{n-1}, a_n) + hm(a_{n-1}, a_n)]$ (4)

Now we have,

$$M(a_{n-1}, a_n) = d(a_{n-1}, a_{n+1}) \leq d(a_{n-1}, a_n) + d(a_{n-1}, a_n).$$

And

$$\text{m}(a_{n-1}, a_n) = 0$$

Let assume that,

$d(a_{n-1}, a_n) < d(a_n, a_{n+1})$ for some n.

So, by the equations (3) and (4), we will get,

$$d(a_n, a_{n+1}) < 2\alpha d(a_n, a_{n+1}) + 2\beta d(a_n, a_{n+1}) + \gamma d(a_n, a_{n+1}) + 2\xi d(a_n, a_{n+1})$$

$$< (2\alpha + 2\beta + \gamma + 2\xi) d(a_n, a_{n+1})$$

$$= d(a_n, a_{n+1})$$

Now, this is our contradiction.
Therefore

$$d(a_{n-1}, a_n) \geq d(a_n, a_{n+1}) \; \forall \, n \qquad (5)$$

By equation (1), we have,

$$d(a_1, fa_2) = d(fa_0, fa_2)$$

$$d(a_1, fa_2) \leq \alpha[d(a_0, fa_0) + d(a_2, fa_2)] + \beta[d(a_0, fa_2) + d(a_2, fa_0)] + \gamma d(a_0, a_1) + \xi\, [M(a_0, a_2) + hm(a_0, a_2)]$$

By the triangle inequality from equation (5), we get

$$d(a_1, fa_2) \leq 2\alpha d(a_0, a_1) + 4\beta d(a_0, a_1) + 2\gamma d(a_0, a_1) + \xi(3 + h) d(a_0, a_1)$$

Then,

$$d(a_1, fa_2) \leq [2(1 - \gamma) - \xi(1 - h)]\, d(a_0, a_1) \qquad (6)$$

Now by equation (1) we have,

$$d(fa_1, fa_2) \leq \alpha'[d(a_1, fa_1) + d(a_2, fa_2)] + \beta'[(a_1, fa_2) + d(a_2, fa_1)] + \gamma' d(a_1, a_2) + \xi'[M(a_1, a_2) + hm(a_1, a_2)]$$

Here

$$d(a_1, fa_2) = M(a_1, a_2) \; and \; m(a_1, a_2) = 0$$

$\alpha', \beta', \gamma', \xi'$ *are nothing but* $\alpha, \beta, \gamma, \xi$ *in a manner that* $2\alpha', 2\beta', \gamma', 2\xi' = 1$

Then by equation (5) and (6), we have,

$$d(fa_1, fa_2) \leq 2\alpha' d(a_0, a_1) + \beta' d(a_1, fa_2) + \gamma' d(a_0, a_1) + \xi' d(a_1, fa_2)$$

$$\leq (2\alpha' + \gamma') d(a_0, a_1) + (\beta' + \xi')\, d(a_1, fa_2)$$

$$\leq (2\alpha' + \gamma') d(a_0, a_1) + (\beta' + \xi')[(2(1 - \gamma) - \xi(1 - h)] d(a_0, a_1)$$

$$d(fa_1, fa_2) \leq 2\alpha' + \gamma' d(a_0, a_1) + 2\beta' - 2\beta'\gamma - \beta'\xi(1 - h) - 2\xi' h - \xi\xi'(1 - h) + 2\xi'$$

$$d(fa_1, fa_2) \leq \left(1 - s^2(1 - h)\right) d(a_0, a_1), where \; s^2 = \xi\xi' \qquad (7)$$

Again, we have from equation (5) and (7),

$$d(fa_2, fa_3) \leq d(fa_1, fa_2)$$

$$\leq (1 - s^2(1 - h)) d(a_0, a_1)$$

And from equation (1) and (5), we have,

$$d(fa_2, fa_4) \leq \alpha[d(a_2, fa_2) + d(a_4, fa_4)] + \beta[(a_2, fa_4) + d(a_4, fa_2)] + \gamma d(a_2, a_4) + \xi[M(a_2, a_4)] + hm(a_2, a_4)]$$

$$\leq 2\alpha d(a_2, a_3) + 4\beta d(a_2, a_3) + 2\gamma d(a_2, a_3) + \xi(3 + h) d(a_2, a_3)$$

$\leq (2(1-\gamma)-\xi(1-h))d(a_0, fa_0)$ (8)

We get from equation (5) and (8),

$$d(fa_3, fa_4) \leq \alpha'[d(a_3, fa_3) + d(a_4, fa_4)] + \beta'[(a_3, fa_4) + d(a_4, fa_3)] + \gamma' d(a_3, a_4) + \xi[M(a_3, a_4)] + hm(a_3, a_4)$$
$$\leq (2\alpha' + \gamma')d(a_3, a_4) + (\beta' + \xi')d(a_3, fa_4)$$

$$\leq (2\alpha' + \gamma')d(a_3, a_4) + (\beta' + \xi')(2(1-\gamma) - \xi(1-h))\, d(a_2, a_3) \leq \left(1 - s^2(1-h)\right)^2 d(a_0\,, a_1)$$

Furthermore,

$$d(fa_4\,, fa_5) \leq d(f(a_3\,, fa_4)$$

$$\leq \left(1 - s^2(1-h)\right)^2 d(a_0\,, a_1)$$

where $\gamma' \geq 0, \alpha', \beta', \gamma' > 0, 2\alpha' + 2\beta' + \gamma' + 2\xi' = 1 \; and \; s^2 = \xi\xi'$

Correspondingly,

$$d(fa_5\,, fa_6) \leq \left(1 - s^2(1-h)\right)^3 d(a_0\,, a_1)$$

And then continue this process
For n term, we have,

$$d(f^n a_0\,, f^{n+1} a_0) \leq \left(1 - s^2(1-h)\right)^{\left(\frac{n}{2}\right)} d(a_0\,, fa_1) \quad (9)$$

Here $(\frac{n}{2})$ indicates the greatest integer which is not exceeding by $(\frac{n}{2})$.

Since, $0 < s < \frac{1}{2}$ and $h < 1$, we have $\lim_{n\to\infty} d(f^n a_0, f^{n+1} a_0) = 0$

Therefore f is asymptotically regular at a_0.

Preposition 2: If f is a self – mapping on x which satisfying the condition (1) with (2) and (3). If f has a fixed point say q then f is continuous at q.

Proof: Let $a_n \to q = T$ then, by triangle inequality which we used in (1) condition and using (3) condition, we get,

$$d(fa_n, fq) \leq \alpha[d(a_n, fa_n) + d(q, fq)] + \beta[(a_n, fq) + d(q, fa_n)] + \gamma d(a_n, q) + \xi\,[M(a_n, q) + hm(a_n, q)]$$

$$\leq \alpha[d(a_n, q) + d(q, fa_n)] + \beta[d(a_n, q) + d(q, fa_n)] + \gamma d(a_n, q) + \xi\,[\max\{d(a_n, q), d(q, fa_n)\} + h\min\{d(a_n, q), d(q, fa_n)\}]$$

$$< (2\alpha + 2\beta + \gamma + 2\xi)d(a_n, q)$$

$$= d(a_n, q)$$

Now $fx_n \to f_q$ whenever $n \to \infty$ therefore f is continuous at q.

Theorem1: Suppose (X, d) be a non-empty complete metric space, and let f be a self-mapping on X which satisfy the following condition (1) with (2) and (3). Then f has a unique fixed point.

Proof: Suppose $a_0 \epsilon X$ be arbitrary. Define a sequence $\{a_n\}$ in X such that $a_{n+1} = f^n x_0$. Then from equation (9) we get the result $\{f^n x_0\}$ is a Cauchy sequence. Since X is complete, there is $q \in X$ such that

$$\lim_{n\to\infty} f^n x = q$$

Now from (1) equation, we have,

$$d(f^n a, fq) \leq \alpha[d(f^{n-1}a, f^n a) + d(q, fq)] + \beta[(f^{n-1}a, fq) + d(q, f^n a)] + \gamma d(f^{n-1}a, q) + \xi\,[M(f^{n-1}a, q) + hm(f^{n-1}, q)]$$

Now using the limit as $n \to \infty$

$$d(q, fq) \le (\alpha + \beta + + \xi) d(q, fq)$$

Then $q = f_q$
Now suppose that q and r are two fixed point then by equation (1), we get,

$$d(q,r) = d(fq, fr) \le \alpha[d(q,fq) + d(r,fr)] + \beta[d(q,fq) + d(r,fr)] + \gamma[d(q,r)] + \xi[M(q,r) + hm(q,r)]$$

$$= (2\beta + \gamma + \xi(1+h)) d(q,r)$$

By using equation (3), we get

$$d(q,r) \le [1 - (2\gamma + \xi(1-h))]\, d(q,r)$$

which implies $q = r$. Hence f has a unique fixed point.

Example1: $Let\ X = \{0,1,3\}. Define\ T : X \to X\ as$:

$T(0) = 0, T(1) = 0, T(3) = 1$

The metric d on X is defined by:

- $d(0,0) = d(1,1) = d(3,3) = 0,$
- $d(0,1) = d(1,0) = 3,$
- $d(0,3) = d(3,0) = 2,$
- $d(1,3) = d(3,1) = 2.$

$$Then, T\ satisfies\ all\ the\ conditions\ (1) for\ \alpha = \frac{3}{10}, \beta = \frac{2}{10}, \gamma = \frac{3}{10}, \xi = \frac{2}{10}, and\ 0\ is\ the\ only\ fixed\ point\ of\ T.$$

$However, T\ does\ not\ satisfy\ the\ condition\ of\ the\ result\ of\ (12)\ for\ x = 0, y = 3.$

2.2 Coincidence point theorem for A Pair of single valued mappings
Theorem 2: Suppose g and f be self- mapping on a metric space (M, d) such that $g(x) \subset f(x)$. Suppose that $a, b \in X$ then the following condition hold.

$$d(ga, gy) \le \delta_1[d(fa, ga) + d(fb, gb)] + \delta_2[d(fa, gb) + d(fb, ga)] + \delta_3 d(fa, fb) + \delta_4 F1[\min\{d(fa, ga), d(fb, gb)\}] + \delta_5 F2[d(fa, ga) d(fb, gb)]$$

where for $i = 1,2,3,4,5, \delta_i > 0$ such that for every arbitrary fixed $k > 0, 0 < \lambda_1 < 1\ and\ 0 < \lambda_2 < 1$, the following conditions hold:

$$\delta_1 + \delta_2 < \lambda_1, \delta_3 + 2\delta_2 < \lambda_2, \delta_4 + \delta_5 < k$$

If $f(M)$ or $g(M)$ is a complete subspace of $M\ and\ g$ is asymptotically $f - regular$ at some point $a_0 \epsilon M$, then g and f have a coincidence point.

Theorem 3: Suppose g and f be self mapping on a metric space (X, d) satisfying for $a, b \epsilon X$, then the condition

$$d(ga, gb) \le \alpha[d(fa, ga) + d(fb, gb)] + \beta[d(fa, gb)] + d(fb, ga)] + \gamma d(fa, fb) + \xi[M_f(a, b) + hm_f(a, b)] \quad (10)$$

Where
$\gamma \ge 0, \alpha, \beta, \xi > 0, and\ 0 < h < 1\ with\ the\ condition\ 2\alpha + 2\beta + \gamma + 2\xi = 1$

$$M_f(a, b) = \max\{d(fa, gb), d(fb, ga)\}\ and\ m_f(a, b) = \min\{d(fa, gb), d(fb, ga)\}$$

If $g(X) \subset f(X)\ and\ g\ is\ asymptotically\ f - regular\ at\ some\ a_0$ in X and one of the following holds
- f is surjective and X is complete
- f is continuous, X is complete and $f\ and\ g$ are compatible

- $f(X)$ is complete subspace of X
- $g(X)$ is complete subspace of X

So, there exist a coincidence point in X for f and g. Furthermore, the coincidence value is unique that is $f_p = f_q$, whenever, $f_p = g_q$ and $f_q = g_q$

Proof: Suppose $a_0 \epsilon X$ be an arbitrary. Since $g(X) \subset f(X)$, we may create a sequence $fx_{n+1} = gx_n$for n ϵ N ∪ {0}. Now from equation (10), we obtain $(h = 1)$

$$d(fa_{n+1}, fa_{n+2}) \leq \alpha[d(fa_n, ga_n) + d(fa_{n+1}, ga_{n+1})] + \beta[d(fa_n, ga_{n+1})] + d(fa_{n+1}, ga_n)] + \gamma d(fa_n, fa_{n+1}) + \xi[M_f(a_n, a_{n+1}) + m_f(a_n, a_{n+1})]$$

The result of above inequality for some n as

$$d(fa_n, fa_{n+1}) < d(fa_{n+1}, fa_{n+2})$$

$$d(fa_{n+1}, fa_{n+2}) < (2\alpha + 2\beta + \gamma + 2\xi)\, d(fa_{n+1}, fa_{n+2})$$

$$= d(fa_{n+1}, fa_{n+2})$$

That is a contradiction. So,

$$d(fa_{n+1}, fa_{n+2}) \leq d(fa_n, fa_{n+1})\ \forall\ \text{n}\ (11)$$

For $m > n$, we obtain

$$d(fa_n, fa_m) \leq d(fa_{n+1}, fa_{m+1}) + d(fa_{m+1}, fa_m) + d(fa_n, fa_{n+1})$$

$$= d(fa_{m+1}, fa_m) + d(ga_n, ga_m) + d(fa_n, fa_{n+1})$$

By triangle inequality and by equation (10) & (11), we obtain,

$$d(fa_n, fa_m) \leq d(fa_{m+1}, fa_m) + d(fa_n, fa_{n+1}) + \alpha[d(fa_n, ga_n) + d(fa_m, ga_m)] + \beta[d(fa_n, ga_m)] + d(fa_m, ga_n)] + \gamma d(fa_n, fa_m) + \xi[M_f(a_n, a_m) + m_f(a_n, a_m)]$$

$$d(fa_n, fa_m) \leq d(fa_{m+1}, fa_m) + d(fa_n, fa_{n+1}) + \alpha[d(fa_n, fa_{n+1}) + d(fa_m, fa_{m+1})] + \beta[d(fa_n, fa_m)] + d(fa_m, fa_{m+1}) + d(fa_m, fa_n) + d(fa_n, fa_{n+1})] + \gamma d(fa_m, fa_n) + \xi[\left(d(fa_n, fa_m) + d(fa_m, fa_{m+1})\right) + \left(d(fa_m, fa_n) + d(fa_m, fa_{m+1})\right)]$$

$$= (2\beta + \gamma + 2\xi) d(fa_n, fa_m) + \gamma[d(fa_n, fa_{n+1}) + d(fa_m, fa_{m+1})]$$

$$= (1 + \beta + \xi)[d(fa_m, fa_{m+1}) + d(fa_n, fa_{n+1})]$$

$$d(fa_n, fa_m) = \left(\frac{1 + \beta + \xi}{2\alpha}\right)[d(fa_m, fa_{m+1}) + d(fa_n, fa_{n+1})]$$

Since g is asymptotically $f - regular$. In the above inequality the right-hand side of equation tends to zero, as $m, n \to \infty$ then, $\lim_{m,n\to\infty} d(fa_n, fa_m) = 0$. This implies that $\{fa_n\}$ is a Cauchy sequence in X. Now, we examine the following cases:

Consider f is surjective and X is complete. Then there exists a point $q \epsilon X$ such that $\lim_{n\to\infty} fa_n = q$ and a point $z \epsilon X$ satisfying $q = fz$. From equation (10), it follows that the result holds with $h = 1$

$$d(gz, fz) \leq d(ga_n, gz) + d(fz, fa_{n+1})$$

$$d(gz, fz) \leq \alpha[d(fa_n, ga_n) + d(fz, gz)] + \beta[d(fa_n, gz) + d(fz, ga_n)] + \gamma d(fa_n, fz) + \xi[M_f(a_{n+1}, z) + m_f(a_{n+1}, z)] + d(fz, fa_{n+1})$$

$$d(gz, fz) \leq \alpha[d(fa_n, fa_{n+1}) + d(fz, gz)] + \beta[d(fa_n, gz) + d(fz, fa_{n+1})] + \gamma d(fa_n, fz) + \xi[max\{d(fa_n, fa_{n+1}), d(fz, gz)\} + min\{d(fa_n, fa_{n+1}), d(fz, gz)\} + d(fz, fa_{n+1})]$$

Letting $n \to \infty$, we obtain

$d(gz, fz) \leq (\alpha + \beta + \xi)(gz, fz)$

which implies that $q = fz = gz$. Therefore, f and g share a coincidence point.

Assume that f is continuous and X is complete, and g and f are compatible. Then,

$$\lim_{n\to\infty} fa_n = q$$

Implies

$$\lim_{n\to\infty} ffa_n = fq$$

From equation (10), with $h = 1$, we obtain,

$$d(gq, fq) \leq d(fga_n, gq) + d(fq, ffa_{n+1})$$

$$d(gq, fq) \leq d(fga_n, gfa_n) + d(gfa_n, gq) + d(fq, ffa_{n+1})$$

$$d(gq, fq) \leq d(fga_n, gfa_n) + \alpha[d(ffa_n, gfa_n) + d(fq, gq)] + \beta[d(ffa_n, gq) + d(fq, gfa_n)] + \gamma d(ffa_n, fq) + \xi[M_f(fa_n, q) + m_f(fa_n, q)] + d(fq, ffa_{n+1})$$

Since,

$$\lim_{n\to\infty} fa_n = \lim_{n\to\infty} g\, a_n = q$$

And g and f are appropriate, it follows that

$$\lim_{n\to\infty} d(fga_n, gfa_n) = 0$$

Using the limit as $n \to \infty$ in the above inequality results in:

$$d(gq, fq) \leq (2\alpha + 2\beta + 2\xi)d(gq, fq)$$

Then

$$d(gq, fq) \leq (1 - \gamma)d(gq, fq)$$

This implies that, $gq = fq$

- X be a complete subspace of $f(X)$, then $q \in f(X)$. Let $z \in f^{-1}q$. In that case, $q = fz$, and the proof follows from case (i).
- If $g(X)$ is a complete subspace of X, then $q \in g(X) \subset f(X)$, and the proof follows from case (iii).

To demonstrate uniqueness, assume q is another coincidence point of f and g, that is $gq = fq$, $gp = fp$.

From equation (10), we have:

$$d(gq, gp) \leq \alpha[d(fq, gq) + d(fp, gp)] + \beta[d(fq, gp)] + d(fp, gq)] + \gamma d(fq, fp) + \xi[M_f(q, p) + m_f(q, p)]$$
$$d(gq, gp) \leq (\gamma + 2\beta + 2\xi)\, d(gq, gp)$$

Then,

$$d(gq, gp) \leq (1 - 2\alpha)d(gq, gp),$$

which implies $g(q) = g(p)$, and consequently, $f(q) = f(p)$.

Remark 2: By setting f as the identity mapping on X ($f = I$) in Theorem 3, we obtain another version of Theorem 1 for $h = 1$.

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