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A comparative study of mappings in metric space and controlled metric space

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Abstract

The objective of this paper is to present a comparative study of mapping in Metric Space and Controlled Metric Space. The study provides the structure, gap analysis and application of Metric Space and Controlled Metric Space**.** A Comparative Study of Mappings in Metric Space and in Controlled Metric Space is done with the help of studying the concept of metric space, its various types of mappings. Following this, a further conceptualization of Controlled Metric Space with its various types of mappings and its applications is done. A Controlled Metric Space is a specialized concept in Mathematics that is used to deal with issues related to the "control" of distances and the behavior of sequences or functions within a metric space. And finally we compare various mappings in metric space with various mappings in controlled metric spaces. In this way we find comparative results of various types of mappings. Thus the paper focuses on comparative study between metric spaces along its properties with controlled metric space and its properties.

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Introduction

A metric space is a fundamental concept in mathematics, particularly in the field of topology and analysis. It is a set equipped with a function that defines a distance between any two elements in the set. A controlled metric space is a specialized concept in mathematics that is used to deal with issues related to the "control" of distances and the behavior of sequences or functions within a metric space. This concept often comes up in the study of coarse geometry, large-scale geometry, and geometric group theory.

Today, metric spaces are a fundamental concept in both pure and applied mathematics. They are used in various fields such as computer science (ex. algorithms and data structures), physics (ex. general relativity) and economics (ex. game theory).

The development of metric spaces was a crucial step in the abstraction and generalization of mathematical concepts, allowing mathematicians to extend ideas of distance, convergence and continuity to a wide variety of contexts. This has enabled significant advances across numerous domains of science and engineering contraction. In 1989, Bakhtin introduced an extension of metric spaces, called b−metric spaces, where many interesting fixed point results for some contractive mappings in b−metric spaces were studied. Also, in 1993, Czerwik extended the results of b-metric spaces. In 2018, Shatanawi *et al*. introduced the $\alpha-\psi$-contraction on the extended b−metric spaces. Recently, many research studies were conducted on b-metric space under different contraction conditions. After that, many authors used $\alpha-\psi$-contraction mapping on different metric spaces. In 2017, Kamran *et al*. presented a very interesting generalization of the b−metric spaces, called extended b-metric spaces. An extension of the extended b-metric spaces, called controlled metric type space, was introduced by Mlaiki *et al*. In this paper, we generalize the results of Mehmet and Mukheimer by introducing the $\alpha-\psi$-contractive mapping on controlled metric type spaces.

Definition

A controlled metric space is typically defined in the context of coarse geometry, where the focus is on the large-scale structure of the space rather than its small-scale details. In these contexts, the notion of control often involves functions that regulate how distances in the space are.

manipulated or measured. A controlled metric space is a metric space where distances and maps are regulated by certain control functions. This concept is particularly useful in coarse geometry, geometric group theory and related fields, where understanding the large-scale properties of spaces and their maps is crucial. The notion of control functions allows for flexibility in how distances are manipulated and provides a framework for studying spaces up to coarse equivalence.

A metric space (X, d) consists of the following components:

A set X: This is the set of elements that we are considering. The elements of X can be numbers, points in a plane, functions, etc.

A metric d: This is a function d: X×X→R (R denotes the set of real numbers) that assigns a real number to each pair of elements in X. This function d is called a distance function or simply a metric, and it must satisfy the following properties for all x, y, z ∈ X:

Non-negativity: d(x, y) ≥0 (the distance between any two points is non-negative).

Identity of indiscernible: d(x, y) = 0 if and only if x=y (the distance between two distinct points is positive, and the distance from a point to itself is zero).

Symmetry: d(x, y) =d(y, x) (the distance from x to y is the same as the distance from y to x).

Triangle inequality: d(x, z) ≤d(x, y) +d(y, z) (the distance between two points x and z is at most the sum of the distances from x to y and from y to z).

Examples

Euclidean Space: One of the most familiar examples is the Euclidean space Rn with the Euclidean distance. For x=(x1, x2… xn) and y= (y1,y2,…,yn) in Rn, the Euclidean distance is given by:

$d(x, y) =(x1-y1)^2+(x2-y2)^2+\cdots+(xn-yn)^2$

Discrete Metric Space: For any set X the discrete metric is defined by:

d(x, y)=0 if x = y

=1 if x≠ y

This metric simply indicates whether two points are the same or different, with no finer gradation of distance.

Manhattan Distance: Also known as the taxicab or L1 distance, in Rn it is defined as: $d(x, y)=|x1-y1|+|x2-y2|+\cdots+|xn-yn|$

Metric spaces provide a framework for discussing and analyzing concepts of distance, convergence, continuity and compactness in a very general setting, applicable in various areas of mathematics and its applications.

Controlled Metric Space Graph: This term can be interpreted in different ways depending on the context. Generally, it could refer to a graph that is constructed or analyzed within a metric space with certain controls or constraints applied. This could involve controlling distances, optimizing paths, or imposing specific rules on how edges are added or weighted based on the metric.

Definition of controlled metric space graph

A controlled metric space graph G= (V, E) within a metric space (M, d) is a graph where:

- Vis a set of vertices corresponding to points in M.
- E⊆V×V is a set of edges with weights determined by the metric d.
- Controls are applied to the edges E, such as:
- An edge (u, v) exists if and only if d (u, v) ≤ δd (u, v) for some threshold δ.
- Edge weights are defined as w (u, v) =f (d (u, v)) for some function f.

Concept of extended b-metric spaces was initiated by Kamran *et al.* in 2017, and their work generalized many results in the literature.

Definition 1: Let Y be an nonempty set and define the mappings

v: Y × Y ⟶ [1,∞) and $\wp$: Y × Y ⟶ [0,∞), such that ∀g, h, r ∈ Y,

- $\wp\,(g, h) = 0 \Leftrightarrow g = h$.
- $\wp\,(g, h) = \wp\,(h, g)$.
- $\wp\,(g, h) \leq]\,(g, h)\,[\wp(g, r) + \wp(r, h)]$. When, we say that the pair (Y, $\wp$) is an extended b−metric space

Types of Mappings in Metric Spaces Isometry

An isometry is a mapping between metric spaces that preserves distances exactly. Formally, a function f: (X, dX) →(Y, dY) is an isometry if for all x, y ∈ X

dY(f(x),f(y))=dX(x,y).

Isometries preserve the structure of the metric space, meaning that the image of an isometry retains the same distances and hence the same geometric and topological properties.

Continuous Mapping

A mapping f:(X,dX)→(Y,dY) is continuous if for every ϵ>0 there exists a δ>0 such that for all x,y∈X,

$dX(x,y)<\delta \Longrightarrow dY(f(x),f(y))<\epsilon$.

Continuity ensures that small changes in the input lead to small changes in the output, preserving the closeness of points under the mapping.

Contraction Mapping

A contraction mapping is a Lipschitz mapping with K<1. Formally, f:(X,dX)→(Y,dY) is a contraction if there exists a constant 0≤K<1 such that for all x, y ∈ X.

$dY(f(x),f(y))\leq K\, dX(x,y)$.

Contraction mappings are significant due to the Banach fixed-point theorem, which guarantees that a contraction mapping on a complete metric space has a unique fixed point.

Homeomorphism
Homeomorphism is a bijective continuous mapping whose inverse is also continuous. Formally, $f:(X,dX)\to(Y,dY)$ is a homeomorphism if f is continuous, bijective, and f is continuous, bijective and $f-1$ is continuous.
Homeomorphisms preserve topological properties, meaning that the spaces are topologically equivalent.

Applications of Mappings in Metric Spaces Analysis and Functional Analysis
In functional analysis, mappings between spaces of functions are studied using metric space concepts. For instance, linear operators between Banach spaces (complete normed vector spaces) are analyzed using continuity and boundedness criteria.

Geometry and Topology
Isometries are crucial in geometry for understanding congruence and similarity of shapes. Homeomorphisms are used in topology to classify spaces up to topological equivalence.

Dynamical Systems
Contraction mappings are used in the study of dynamical systems to prove the existence and uniqueness of equilibrium points or stable states.

Computer Science
In algorithm design, especially in optimization and machine learning, Lipschitz continuity can be used to bound the behavior of functions and ensure convergence properties of algorithms.

Fixed-Point Theorems
Contraction mappings lead to important fixed-point theorems such as the Banach fixed-point theorem, which has applications in various fields including economics, game theory, and differential equations.
Mappings in metric spaces allow mathematicians and scientists to understand how different spaces relate to one another and how various properties are preserved or transformed under these mappings. They are fundamental in both theoretical investigations and practical applications across many disciplines.

Types of Mappings in Controlled Metric Spaces
Controlled Mapping
A mapping $f:(X,dX)\to(Y,dY)$ between two metric spaces is called a controlled mapping if there exists a control function $\rho:R+\to R+$ such that for all $x,y\in X$, $dY(f(x),f(y))\le\rho(dX(x,y))$. This definition ensures that the distance in the target space Y is bounded by a function of the distance in the source space X. The function ρ dictates how much distortion is allowed.

Lipschitz Mapping
A special case of a controlled mapping is a Lipschitz mapping, where the control function is linear. A function $f:(X,dX)\to(Y,dY)$ is Lipschitz if there exists a constant $K\ge0$ such that for all $x,y\in X$

$dY(f(x),f(y))\le K\ dX(x,y)$.

if k=1 the mapping is non expansive

if $0<=k<=1$ the mapping is contraction. K=1, the mapping is Non-expansive.

Coarse Mapping
A coarse mapping between two metric spaces (X,dX) and (Y,dY) is a mapping $f:X\to Y$ for which there exist functions $\rho1$, $\rho2$: $R+\to R+$ such that for all $x,y\in X$

$\rho1(dX(x,y))-C\le dY(f(x),f(y))\le\rho2(dX(x,y))+C$.

This type of mapping is more general than controlled or Lipschitz mappings, allowing both upper and lower bounds on how distances can be distorted.

Quasi-Isometry
A quasi-isometry is a specific type of coarse mapping where the distortion is controlled both above and below by linear functions. A map $f:X\to Y$ is a quasi-isometry if there exist constants $A\ge1$ and $B\ge0$ such that for all $x, y \in X$, and every point in Y is within a bounded distance (depending on B) of some point in the image of f.

Controlled Homotopy
Two mappings $f,g:(X,dX)\to(Y,dY)$ are controlled homotopic if there exists a continuous mapping $H:X\times[0,1]\to Y$ (called a homotopy) such that for each $t\in[0,1]$ the mapping $Ht:X\to YH\_$ defined by $Ht(x)=H(x,t)$ is controlled with a uniform control function ρ.
This means that the homotopy does not distort distances excessively at any stage.

Applications and Significance
Mappings in controlled metric spaces are used in various mathematical fields, including:

Coarse Geometry
Coarse mappings, particularly quasi-isometries, are central in coarse geometry, where the focus is on properties that are invariant under large-scale transformations. This is essential in understanding spaces that may look different at small scales but are similar when viewed from a large-scale perspective.

Geometric Group Theory
Quasi-isometries are used to study groups through their Cayley graphs. Two finitely generated groups are quasi-isometric if their Cayley graphs are quasi-isometric. This helps in classifying groups based on their large-scale geometric properties.

Topology
Controlled homotopies and coarse mappings help in studying large-scale topological properties of spaces. This is useful in areas like topological data analysis, where the shape of data at large scales is of interest.

Analysis on Metric Spaces
Controlled mappings are relevant in the study of functions and operators on metric spaces, particularly in functional analysis and potential theory.
Let X be a nonempty set and $\Omega: X\times X\to[1,\infty)$ be a given

function. An extended b-metric is a function $d : X \times X \to [0, \infty)$ such that for all $\eta, \xi, u \in X$,

1. $d(\eta, \xi) = 0 \Longleftrightarrow \eta = \xi$;

2. $d(\eta, \xi) = d(\xi, \eta)$;
3. $d(\eta, \xi) \leq \theta(\eta, \xi)[d(\eta, u) + d(u, \xi)]$.

One of generalizations of b−metric spaces has been provided by Mlaiki, who introduced the concept of controlled metric type spaces by employing a control function $\delta : X \times X \to [1, \infty)$ to act separately on each term in the right-hand side of the triangle inequality.

In summary, mappings in controlled metric spaces allow mathematicians to understand and manipulate these spaces while preserving certain controlled distortions of distances. This is crucial in various mathematical and applied contexts, especially those dealing with large-scale structures and properties.

Table 1: Comparison between metric space and controlled metric space

Feature	Metric Space	Controlled Space
Definition	A set with a distance function (metric) that defines the distance between any two elements.	A system in which inputs (controls) influence the state of the system to achieve desired outcomes.
Key Components	Set of points\n- Metric (distance function)	System dynamics\n- Control inputs\n- Desired outcomes
Metric	A function $d:X\times X\to R$ satisfying non-negativity, identity of indiscernibles, symmetry, and triangle inequality.	Not applicable (focus on control inputs and outputs rather than distances)
Primary Purpose	To measure the distance between elements in a set.	To manipulate the behavior of a system to achieve specific goals.
Example	Euclidean space with the Euclidean distance.	A thermo stat controlling the temperature of a room.
Applications	Geometry, topology, analysis.	Engineering, robotics, economics, and any field requiring system regulation.
Distance/Control Function	Distance between points in the set.	Control laws or strategies that govern system behavior.
Properties	Non-negativity, identity of indiscernibles, symmetry, triangle inequality.	Stability, controllability, observability, and optimality.
Structure	Typically focuses on abstract properties of sets and functions.	Involves physical or abstract systems and the dynamics governing their behavior.

This table highlights the fundamental differences and applications of metric spaces and controlled spaces.

Graphical Comparison

Graphical representations can provide intuitive insights into the differences between metric spaces and controlled metric spaces.

Visualizing Continuity

Metric Space Example

Graph $f(x)=x^2$ on the real line with the standard metric $d(x,y)=|x-y|$. Plot the function and its behavior for various δ\ϵ.

Controlled Metric Space Example

Suppose $\rho(x,y)=|x-y|$ (controlled by $\varphi(t)=t$). Compare how changes in δ affect the continuity in this metric compared to the standard metric.

Graphical Representation

Metric Space: Plot $f(x)=x^2$and ϵ\δ regions.

Controlled Metric Space: Plot $f(x)=x^2$ under ρ and compare how ϵ\ϵ-δ\δ regions change.

Visualizing Contractions and Fixed Points

Metric Space Example: Plot $f(x)=x^2$ in the unit interval. Show how the Banach fixed-point theorem applies.

Controlled Metric Space Example

For $\rho(x,y)=|x-y|^2$, plot $f(x)=x^2$ and visualize fixed points and convergence.

Graphical Representation

Metric Space

Show the contraction mapping on a plot with fixed points.

Controlled Metric Space

Compare the fixed-point behavior with $\rho(x,y)=|x-y|^2$.

Visualizing Isometries

Metric Space Example

For $f(x)=x+1$ in R, show that it is not an isometry under $d(x,y)=|x-y|$.

Controlled Metric Space Example

If $\rho(x,y)=|x-y|$, then $f(x)=x$ is an isometry.

Graphical Representation

Metric Space

Show the distance preservation for $f(x)=x$ compared to $f(x)=x+1$.

Controlled Metric Space

Compare distance preservation for different control functions.

Compactness and Completeness

Metric Space Example: Plot sequences converging in R and show completeness.

Controlled Metric Space Example: If $\rho(x,y)=|x-y|$, visualize sequences converging under **ρ** and compare with the standard metric.

Graphical Representation

Metric Space: Show Cauchy sequences and their limits.

Controlled Metric Space: Show how controlled metrics affect Cauchy sequences and completeness.

Comparative Analysis Summary

- **Mathematical Insights:** Compare properties such as continuity, fixed points, isometries, and completeness between standard and controlled metrics. Highlight how control functions affect these properties.
- **Graphical Insights:** Use plots to illustrate differences in behavior under standard and controlled metrics. Provide visual examples to show how metrics impact properties and mappings.

Result analysis

Comparative Analysis between metric and controlled space:

Existence of Fixed Points: Metric Spaces

The Banach fixed-point theorem ensures the existence and uniqueness of fixed points for contraction mappings in complete metric spaces.

Controlled Metric Spaces

Fixed-point results might be adapted or generalized based on the specific control conditions. For example, the control parameters might affect the contraction constant or completeness conditions.

Compactness and Completeness

Metric Spaces

Compactness and completeness are well-understood properties in metric spaces. Compactness often leads to powerful results like the Heine-Borel theorem.

Controlled Metric Spaces

The notions of compactness and completeness might be adjusted depending on the control mechanisms. For example, the controlled metric might lead to different compactness criteria.

Convergence and Stability

Metric Spaces

In the context of iterative processes, the convergence of sequences and stability of solutions are well-established.

Controlled Metric Spaces

Convergence might be influenced by control parameters, and stability analysis might need to account for the additional complexity introduced by the control.

Conclusion

Each type of metric space has its own nature unique properties which are used for different applications. Controlled Metric Spaces are useful for a controlled system. The choice of which metric space to use depends on the specific requirements and nature of the problem at hand. Controlled metric spaces and metric spaces are both fundamental concepts in the field of topology and analysis. Metric Space typically focuses on abstract properties of sets and functions where as controlled metric space focuses or involves physical or abstract systems and the dynamics governing their behavior.

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