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## On the ternary quadratic equation $7(x^2+y^2) - 13xy = 27z^2$

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### Abstract

We obtain infinitely many non-zero integer triples  $(x, y, z)$  satisfying the homogenous quadratic equation with three unknowns. Various interesting properties among the values of  $CC$  and  $z$  are presented. Some relations between the solutions and special numbers are exhibited.

**Keywords:** Ternary quadratic, Homogeneous quadratic, integral solutions, 2010 Mathematics subject classification, 11D09

### Introduction

There is a great interest for mathematicians since ambiguity in homogeneous and non-homogeneous Quadratic Diophantine Equations <sup>[1-3]</sup>. In this context, one may refer <sup>[4-15]</sup> for varieties of problems on the quadratic Diophantine equations with three, four and five variables. In this paper, quadratic equation with three variables given by  $7(x^2 + y^2) - 13xy = 27z^2$  is analyzed for its non-zero distinct integer solutions. A few interesting relations between the solutions and special numbers are exhibited.

### Method of analysis

The Diophantine equation representing the ternary quadratic equation to be solved for its non-zero distinct integral solution is

$$7(x^2 + y^2) - 13xy = 27z^2 \quad (1)$$

Substituting the linear transformations

$$x = 3u + v, y = 3u - v, u \neq v \neq 0 \quad (2)$$

In (1), it leads to

$$u^2 + 3v^2 = 3z^2 \quad (3)$$

(3) Can be solved through different methods and we obtain different sets of integer solutions to (1).

**Set 1**

$$\text{Assume } z = a^2 + 3b^2 \quad (4)$$

Where a and b are non-zero distinct integers.

$$\text{Write 3 as } 3 = (i\sqrt{3})(-i\sqrt{3}) \quad (5)$$

Substituting (4) and (5) in (3) and applying the method of factorization, define

$$u + i\sqrt{3}v = (i\sqrt{3})(a + i\sqrt{3}b)^2$$

Equating the real and imaginary parts, we have

$$\left. \begin{aligned} u &= -6ab \\ v &= a^2 - 3b^2 \end{aligned} \right\} \quad (6)$$

From (2), the non-zero distinct integer solutions to (1) are found to be

$$\begin{aligned} x &= -18ab + (a^2 - 3b^2) \\ y &= -18ab - (a^2 - 3b^2) \\ z &= a^2 + 3b^2 \end{aligned}$$

**Properties**

- $y(2^b, b) + z(2^b, b) + 18(W_b - 1)$  is a nasty number.
- $2(2t_{4,a} - x(a, a) - z(a, a))$  is a perfect square.
- $x(a, a) + y(a, a) + 36t_{4,a} = 0$ .

**Note 1**

Instead of (5), write 3 as

$$3 = \frac{(3 + i\sqrt{3})(3 - i\sqrt{3})}{4}$$

Following the procedure as in set 1, the corresponding non-zero distinct integer solutions to (1) are obtained as

$$\begin{aligned} x &= 5a^2 - 15b^2 - 6ab \\ y &= 4a^2 - 12b^2 - 12ab \\ z &= a^2 + 3b^2 \end{aligned}$$

**Note 2**

Instead of (5), write 3 as

$$3 = \frac{(12 + i\sqrt{3})(12 - i\sqrt{3})}{49}$$

By applying the same process as derived in set 1, we have the corresponding non-zero distinct integer solutions to (1) are obtained as

$$\begin{aligned}x &= 259A^2 - 777B^2 + 42AB \\y &= 245A^2 - 735B^2 - 294AB \\z &= 49(A^2 + 3B^2)\end{aligned}$$

**Set 2**

One may write (3) as

$$u^2 + 3v^2 = 3z^2 * 1 \quad (7)$$

Write 1 as

$$1 = \frac{(1 + i\sqrt{3})(1 - i\sqrt{3})}{4} \quad (8)$$

Using (4), (5) and (8) in (7) and applying the method of factorization, define

$$(u + i\sqrt{3}v) = (i\sqrt{3}) \frac{(1 + i\sqrt{3})}{2} (a + i\sqrt{3}b)^2 \quad (9)$$

Equating the real and imaginary parts of (9), we have

$$\begin{aligned}u &= \frac{1}{2} \{-3a^2 + 9b^2 - 6ab\} \\v &= \frac{1}{2} \{a^2 - 3b^2 - 6ab\}\end{aligned}$$

In view of (2), the integer solutions of (1) are given by

$$\begin{aligned}x &= -4a^2 + 12b^2 - 12ab \\y &= -5a^2 + 15b^2 - 6ab \\z &= a^2 + 3b^2\end{aligned}$$

**Note 3**

- Instead of (11), '1' can also be considered as  $1 = \frac{(1 + i4\sqrt{3})(1 - i4\sqrt{3})}{49}$ , and the corresponding non-zero solutions are as follows:

$$\begin{aligned}x &= -196A^2 + 588B^2 - 3528AB \\y &= -560A^2 + 1680B^2 - 3024AB \\z &= 196(A^2 + 3B^2)\end{aligned}$$

- It is worth to mention that, '1' maybe considered in general as

$$1 = \frac{\{(3r^2 - s^2) + i\sqrt{3}rs\}\{(3r^2 - s^2) - i\sqrt{3}rs\}}{(3r^2 + s^2)^2}$$

**Properties**

- $3\{4z(a, a+1) - x(a, a+1) - 12Pr_a\}$  is a nasty number.

- $5z(1,b) + y(1,b) - 10t_{4,b} - S_b + 1 = 0.$
- $z(a,a) - x(a,a) + 2y(a,a)$  is a perfect square.

**Set 3**

Equation (3), can also be written as  $3z^2 - u^2 = 3v^2$  (10)

Assume  $v = 3a^2 - b^2$  (11)

Write 3 as  $3 = (2\sqrt{3} + 3)(2\sqrt{3} - 3)$  (12)

Substituting (11) and (12) in (10) and applying the method of factorization, define

$$\sqrt{3}z + u = (2\sqrt{3} + 3)(\sqrt{3}a + b)^2$$

Equating the rational and irrational roots, we get

$$\left. \begin{aligned} u &= 9a^2 + 3b^2 + 12ab \\ z &= 6a^2 + 2b^2 + 6ab \end{aligned} \right\} \quad (13)$$

From (2), the non-zero distinct integer solutions to (1) are found to be

$$\begin{aligned} x &= 30a^2 + 8b^2 + 36ab \\ y &= 24a^2 + 10b^2 + 36ab \\ z &= 6a^2 + 2b^2 + 6ab \end{aligned}$$

**Properties**

- $x(a + 1, a) - 5z(a + 1, a) + 2 * t_{4,a} = 6 * Pr_a$ .
- $6z(2^a + 1, 1) - y(2^a + 1, 1) - 12Ky_a - 26 = 0.$
- $6 [x(a, a) - y(a, a)]$  is a nasty number.

**Note 4**

In equation (12), we can choose ‘3’ as  $3 = (7\sqrt{3} + 12)(7\sqrt{3} - 12)$ .

For this choice, we obtain the non-zero distinct integer solution to (1) are given by

$$\begin{aligned} x &= 111a^2 + 35b^2 + 126ab \\ y &= 105a^2 + 37b^2 + 126ab \\ z &= 21a^2 + 7b^2 + 24ab \end{aligned}$$

**Set 4**

Now we are going to introduce another set of transformations as

$$u = X + 3T, \quad v = X - T, \quad z = 2W \quad (14)$$

In (3), we obtain

$$X^2 + 3T^2 = 3W^2 \quad (15)$$

For this, the solutions are said to be

$$\left. \begin{aligned} W &= a^2 + 3b^2 \\ X &= -6ab \\ T &= a^2 - 3b^2 \end{aligned} \right\} \quad (16)$$

By using (16) in (14), we get

$$\begin{aligned} u &= 3a^2 - 9b^2 - 6ab \\ v &= -a^2 + 3b^2 - 6ab \\ z &= 2a^2 + 6b^2 \end{aligned}$$

In view of (2), the non-zero distinct integer solutions to (1) are found to be

$$\begin{aligned} x &= 8a^2 - 24b^2 - 24ab \\ y &= 10a^2 - 30b^2 - 12ab \\ z &= 2a^2 + 6b^2 \end{aligned}$$

### Properties

- $x(a, a) - 2y(a, a)$  is a nasty number.
- $x(1, \sqrt{2^b}) - 2y(1, \sqrt{2^b}) - 12Tk_b = 0$ .
- $\frac{-1}{8} [x(a, a) + 4z(a, a)]$  is a perfect square.

### Notations Used

- Regular Polygonal Number of rank  $n$  with sides  $m$ :  $t_{m,n} = n \left[ 1 + \frac{(n-1)(m-2)}{2} \right]$
- Pronic Number of rank  $n$ :  $Pr_n = n(n+1)$
- Woodall Number of rank  $n$ :  $W_n = n(2^n) - 1$
- Star Number of rank  $n$ :  $S_n = 6n(n-1) + 1$
- Thabit-ibn-Kurrah Number of rank  $n$ :  $TK_n = 3(2^n) - 1$
- Kynea Number of rank  $n$ :  $Ky_n = (2^n + 1)^2 - 2$

### Conclusion

In this paper, we have presented different patterns of integer solutions to the ternary quadratic equation  $7(x^2 + y^2) - 13xy = 27z^2$ . As the diophantine equations are rich in variety, one may attempt to find integer solutions to other choices of equations of degree two with three or more unknowns.

### References

1. Dickson LE. History of Theory of Numbers, vol 2, Chelsea publishing company, New York; c1952.
2. Mordell LJ. Diophantine Equations, Academic press, London; c1969.
3. Carmichael RD. The theory of numbers and Diophantine analysis, New York, Dover; c1959.
4. Gopalan MA, Vidhyalakshmi S, Kavitha A, Marymadona D. On the Ternary Quadratic Diophantine equation  $3(x^2 + y^2) - 2xy = 4z^2$ , International Journal of Engineering science and Management. 2015;5(2):11-18.
5. Meena K, Vidhyalakshmi S, Bhuvaneshwari E, Presenna R. On ternary quadratic Diophantine equation  $5(X^2 + Y^2) - 6XY = 20Z^2$ . International Journal of Advanced Scientific Research. 2016;1(2):59-61.
6. Devibala S, Gopalan MA. On the ternary quadratic Diophantine equation  $7x^2 + y^2 = z^2$ . International Journal of Emerging Technologies in Engineering Research. 2016;4:9.
7. Bharathi N, Vidhyalakshmi S. Observation on the Non-Homogeneous Ternary Quadratic Equation  $x^2 - xy + y^2 + 2(x+y) + 4 = 12z^2$ . Journal of mathematics and informatics. 2017;10:135-140.

8. Priya A, Vidhyalakshmi S. On the Non-Homogeneous Ternary Quadratic Equation  $2(x^2 + y^2) - 3xy + (x + y) + 1 = z^2$ , Journal of mathematics and informatics. 2017;10:49-55.
9. Gopalan MA, Vidhyalakshmi S, Rajalakshmi UK. On ternary quadratic Diophantine equation  $5(X^2 + Y^2) - 6XY = 196Z^2$ . Journal of mathematics. 2017;3(5):1-10.
10. Gopalan MA, Vidhyalakshmi S, Thangam AS. On ternary quadratic equation  $X(X + Y) = Z + 20$ , The International Journal of Innovative Research in Science, Engineering and Technology. 2017;6(8):15739-15741.
11. Gopalan MA, Umar S. On the Hyperbola  $2x^2 - 3y^2 = 23$ ". Journal of Mathematics and Informatics. 2017;10:1-9.
12. Usha Rani TR, Ambika K. Observation on the Non-Homogeneous Binary Quadratic Diophantine Equation  $5x^2 - 6y^2 = 5$  Journal of Mathematics and Informatics. 2017;10:67-74.
13. Vidhyalakshmi S, Sathya A, Nivetha S. On the pellian like Equation  $5x^2 - 7y^2 = -8$  ". The International Journal of Innovative Research in Science, Engineering and Technology. 2019;6(3):979-984.
14. Usha Rani TR, Bahavathi V, Sridevi S. Observations on the Non-homogeneous binary Quadratic Equation  $8x^2 - 3y^2 = 20$  . The International Journal of Innovative Research in Science, Engineering and Technology. 2019;6(3):2375-2382.
15. Vidhyalakshmi S, Mahalakshmi T. A Study On The Homogeneous Cone  $x^2 + 7y^2 = 23z^2$ ". International Research Journal of Engineering and Technology (IRJET). 2019;6(3):5000-5007.