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On the Pell-Like Equation $7x^2$-$5y^2$ = 8

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Abstract

The hyperbola represented by the binary quadratic equation $7x^2$-$5y^2$ = 8 is analyzed for finding its non-zero distinct integer solutions. A few interesting relations among its solutions are presented. Also, knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented.

Keywords: Pell like equation, Binary quadratic, Hyperbola, Parabola

Introduction

The binary quadratic Diophantine equations of the form $ax^2 - by^2 = N, (a, b, N \neq 0)$ are rich in variety and have been analyzed by many mathematicians for their respective integer solutions for particular values of a, b and N. In this context, one may refer [1-12].

This communication concerns with the problem of obtaining non-zero distinct integer solutions to the binary quadratic equation given by $7x^2 - 5y^2 = 8$ representing hyperbola. A few interesting relations among its solutions are presented. Knowing an integral solution of the given hyperbola, integer solutions for other choices of hyperbolas and parabolas are presented.

Method of Analysis

The Pell - like equation representing hyperbola under consideration is

$$7x^2 - 5y^2 = 8 \rightarrow (1)$$

Whose smallest positive integer solutions is

$$T_0 = 2, X_0 = 12$$

To obtain the other solution of (1),

Consider the Pell equation

$$X^2 = 35T^2 + 1$$

Whose general solution is given by

$$\tilde{T}_n = \frac{1}{2\sqrt{35}} g_n$$

$$\tilde{X}_n = \frac{1}{2} f_n$$

Where,

$$f_n = \left(6+\sqrt{35}\right)^{n+1} + \left(6-\sqrt{35}\right)^{n+1}$$

$$g_n = \left(6+\sqrt{35}\right)^{n+1} - \left(6-\sqrt{35}\right)^{n+1}$$

Appling Brahmagupta lemma between (T_0, X_0) and $(\tilde{T}_n, \tilde{X}_n)$, the other integer solutions of (1) are given by

$$x_{n+1} = 11 f_n + \frac{65}{\sqrt{35}} g_n$$

$$y_{n+1} = 13 f_n + \frac{77}{\sqrt{35}} g_n$$

The recurrence relations satisfied by x and y are given by

$$x_{n+1} - 12x_{n+2} + x_{n+3} = 0$$
$$y_{n+1} - 12y_{n+2} + y_{n+3} = 0$$

Some numerical examples of x and y satisfying (1) are given in the following table

Table 1

n	x_{n+1}	y_{n+1}
-1	22	26
0	262	310
1	3122	3694
2	37202	44018
3	443302	524522

From the above table, we observe some interesting relations among the solutions which are presented below:
Both x_{n+1} and y_{n+1} are always even.
Each of the following expressions is a nasty number

$$(i)\quad 65100x_{2n+2} - 5460x_{2n+3} - 1680$$

$$(ii)\quad 9308880x_{2n+2} - 65520x_{2n+4} - 241920$$

$$(iii)\quad 32340x_{2n+2} - 27300y_{2n+2} - 1680$$

$$(iv)\quad 2310840x_{2n+2} - 163800y_{2n+3} - 60480$$

$$(v)\quad 325843140x_{2n+2} - 1938300y_{2n+4} - 8468880$$

$(vi)\quad 775740x_{2n+3} - 65100x_{2n+4} - 1680$

$(vii)\quad 194040x_{2n+3} - 1953000y_{2n+2} - 60480$

$(viii)\quad 385140x_{2n+3} - 325500y_{2n+3} - 1680$

$(ix)\quad 27536040x_{2n+3} - 1953000y_{2n+4} - 60480$

$(x)\quad 2296140x_{2n+4} - 275387700y_{2n+2} - 8468880$

$(xi)\quad 2310840x_{2n+4} - 23272200y_{2n+3} - 60480$

$(xii)\quad 4589340x_{2n+4} - 3878700y_{2n+4} - 1680$

$(xiii)\quad 4620y_{2n+3} - 55020y_{2n+2} - 1680$

$(xiv)\quad 55440y_{2n+4} - 7867440y_{2n+2} - 241920$

$(xv)\quad 55020y_{2n+4} - 655620y_{2n+3} - 1680$

Remarkable Observations

1). Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of hyperbola which are presented in the table below:

Table 2

S.NO	Hyperbola	x, y
1.	$5x^2 - 7y^2 = 80$	$x = 155x_{n+1} - 13x_{n+2}$ $y = 11x_{n+2} - 131x_{n+1}$
2.	$5x^2 - 7y^2 = 11520$	$x = 1847x_{n+1} - 13x_{n+3}$ $y = 11x_{n+3} - 1561x_{n+1}$
3.	$x^2 - 35y^2 = 16$	$x = 77x_{n+1} - 65y_{n+1}$ $y = 11y_{n+1} - 13x_{n+1}$
4.	$x^2 - 35y^2 = 576$	$x = 917x_{n+1} - 65y_{n+2}$ $y = 11y_{n+2} - 155x_{n+1}$
5.	$x^2 - 35y^2 = 80656$	$x = 10927x_{n+1} - 65y_{n+3}$ $y = 11y_{n+3} - 1847x_{n+1}$
6.	$5x^2 - 7y^2 = 80$	$x = 1847x_{n+2} - 155x_{n+3}$ $y = 131x_{n+3} - 1561x_{n+2}$
7.	$x^2 - 35y^2 = 576$	$x = 77x_{n+2} - 775y_{n+1}$ $y = 131y_{n+1} - 13x_{n+2}$
8.	$x^2 - 35y^2 = 16$	$x = 917x_{n+2} - 775y_{n+2}$ $y = 131y_{n+2} - 155x_{n+2}$
9.	$x^2 - 35y^2 = 576$	$x = 10927x_{n+2} - 775y_{n+3}$ $y = 131y_{n+3} - 1847x_{n+2}$
10.	$x^2 - 35y^2 = 80656$	$x = 77x_{n+3} - 9235y_{n+1}$ $y = 1561y_{n+1} - 13x_{n+3}$

11.	$x^2-35y^2=576$	$x=917x_{n+3}-9235y_{n+2}$ $y=1561y_{n+2}-155x_{n+3}$
12.	$x^2-35y^2=16$	$x=10927x_{n+3}-9235y_{n+3}$ $y=1561y_{n+3}-1847x_{n+3}$
13.	$x^2-35y^2=784$	$x=77y_{n+2}-917y_{n+1}$ $y=155y_{n+1}-13y_{n+2}$
14.	$x^2-35y^2=112896$	$x=77y_{n+2}-10927y_{n+1}$ $y=1847y_{n+1}-13y_{n+3}$
15.	$x^2-35y^2=784$	$x=917y_{n+3}-10927y_{n+2}$ $y=1847y_{n+2}-155y_{n+3}$

2). Employing linear combinations among the solutions of (1), one may generate integer solutions for other choices of parabola which are presented in the table below:

Table 3

S.NO	Parabola	x, y
1.	$10x-7y^2=80$	$x=155x_{2n+2}-13x_{2n+3}+4$ $y=11x_{n+2}-131x_{n+1}$
2.	$120x-7y^2=11520$	$x=1847x_{2n+2}-13x_{2n+4}+48$ $y=11x_{n+3}-1561x_{n+1}$
3.	$2x-35y^2=16$	$x=77x_{2n+2}-65y_{2n+2}+4$ $y=11y_{n+1}-13x_{n+1}$
4.	$12x-35y^2=576$	$x=917x_{2n+2}-65y_{2n+3}+24$ $y=11y_{n+2}-155x_{n+1}$
5.	$142x-35y^2=80656$	$x=10927x_{2n+2}-65y_{2n+4}+284$ $y=11y_{n+3}-1847x_{n+1}$
6.	$10x-7y^2=80$	$x=1847x_{2n+3}-155x_{2n+4}+4$ $y=131x_{n+3}-1561x_{n+2}$
7.	$12x-35y^2=576$	$x=77x_{2n+3}-775y_{2n+2}+24$ $y=131y_{n+1}-13x_{n+2}$
8.	$2x-35y^2=16$	$x=917x_{2n+3}-775y_{2n+3}+4$ $y=131y_{n+2}-155x_{n+2}$
9.	$12x-35y^2=576$	$x=10927x_{2n+3}-775y_{2n+4}+24$ $y=131y_{n+3}-1847x_{n+2}$
10.	$142x-35y^2=80656$	$x=77x_{2n+4}-9235y_{2n+2}+284$ $y=1561y_{n+1}-13x_{n+3}$
11.	$12x-35y^2=576$	$x=917x_{2n+4}-9235y_{2n+3}+24$ $y=1561y_{n+2}-155x_{n+3}$
12.	$2x-35y^2=16$	$x=10927x_{2n+4}-9235y_{2n+4}+4$ $y=1561y_{n+3}-1847x_{n+3}$
13.	$2x-5y^2=112$	$x=77y_{2n+3}-917y_{2n+2}+28$ $y=155y_{n+1}-13y_{n+2}$
14.	$24x-5y^2=16128$	$x=77y_{2n+4}-10927y_{2n+2}+336$ $y=1847y_{n+1}-13y_{n+3}$

15.	$2x-5y^2=112$	$x=917y_{2n+4}-10927y_{2n+3}+28$ $y=1847y_{n+2}-155y_{n+3}$

Conclusion

In this paper, we have made an attempt to obtain all integer solutions through a single process. To conclude, One may search for the equations for which the above method is applicable.

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