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On finding integer solutions to Non-homogeneous ternary bi-quadratic equation $x^2 + 3y^2 = 31z^4$

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Abstract

This paper concerns with the problem of obtaining non-zero distinct integer solutions to the non-homogeneous ternary bi-quadratic equation $x^2 + 3y^2 = 31z^4$. Different sets of integer solutions are illustrated.

Keywords: non-homogeneous bi-quadratic, ternary bi-quadratic, integer solutions

1. Introduction

The Diophantine equations are rich in variety and offer an unlimited field for research [1-4]. In particular refer [5-30] for a few problems on Biquadratic equation with 3 unknowns. This paper concerns with yet another interesting Biquadratic Diophantine equation with three variables given by $x^2 + 3y^2 = 31z^4$ for determining its infinitely many non-zero distinct integral solutions

2. Method of Analysis

The non-homogeneous ternary bi-quadratic equation under consideration is

$$x^2 + 3y^2 = 31z^4 \tag{1}$$

To start with, it is seen that (1) is satisfied by the following integer triples:

$$(x, y, z) = (2k^2, 3k^2, k), (8k^2, 12k^2, 2k), (14k^2, 10k^2, 2k), (22k^2, 2k^2, 2k) .$$

However, there are other sets of integer solutions to (1) and we illustrate different ways of Solving (1) below:

Way 1

Let

$$z = a^2 + 3b^2 \tag{2}$$

Write 31 on the R.H.S. of (1) as

$$31 = (2 + i3\sqrt{3})(2 - i3\sqrt{3}) \tag{3}$$

Substituting (2) & (3) in (1) and employing the method of factorization, consider

$$x + i\sqrt{3}y = (2 + i3\sqrt{3})(a + i\sqrt{3}b)^4 \tag{4}$$

On equating the real and imaginary parts in (4), the values of x, y are given by

$$\left.\begin{aligned} x &= 2(a^4 - 18a^2b^2 + 9b^4) - 9(4a^3b - 12ab^3), \\ y &= 3(a^4 - 18a^2b^2 + 9b^4) + 2(4a^3b - 12ab^3) \end{aligned}\right\} \tag{5}$$

Thus, (2) and (5) represent the integer solutions to (1).

Note 1
The integer 31 on the R.H.S. of (1) is also represented as

$$31 = \frac{(11 + i\sqrt{3})(11 - i\sqrt{3})}{4}$$

Repetition of the above process leads to a different set of integer solutions to (1).

Way 2
Rewrite (1) as

$$x^2 + 3y^2 = 31z^4 * 1 \tag{6}$$

Consider 1 on the R.H.S. of (6) as

$$1 = \frac{(1 + i\sqrt{3})(1 - i\sqrt{3})}{4} \tag{7}$$

Following the analysis similar to Way1, the values of x, y are given by

$$\left.\begin{aligned} x &= 63a^2b^2 - 30a^3b + 90ab^3 - \frac{7a^4 + 63b^4}{2}, \\ y &= -45a^2b^2 - 14a^3b + 42ab^3 + \frac{5a^4 + 45b^4}{2} \end{aligned}\right\} \tag{8}$$

As our interest is on finding integer solutions, observe that (2) and (8) represent the integer
Solutions to (1) provided A and B are of the same parity.

Note 2
The integer 1 on the R.H.S. of (6) is also expressed as

$$1 = \frac{(3r^2 - s^2 + i2\sqrt{3}rs)(3r^2 - s^2 - i2\sqrt{3}rs)}{(3r^2 + s^2)^2},$$

$$1 = \frac{(13 + i3\sqrt{3})(13 - i3\sqrt{3})}{196}$$

Repeating the above process, different sets of solutions to (1) are obtained.

Way 3

Express (1) in the form of ratios as

$$\frac{x+2z^2}{3z^2+y}=\frac{3(3z^2-y)}{x-2z^2}=\frac{\alpha}{\beta},\beta\neq 0 \qquad (9)$$

Solving the above system of double equations (9), one has

$$x=2\alpha^2+18\alpha\beta-6\beta^2, y=-3\alpha^2+4\alpha\beta+9\beta^2, z^2=\alpha^2+3\beta^2 \qquad (10)$$

The third equation in (10) is satisfied by

$$\alpha=3p^2-q^2, \beta=2pq \qquad (11)$$

And

$$z=3p^2+q^2 \qquad (12)$$

Substituting (11) in the first two equations of (10), the values of x, y are given by

$$\left.\begin{array}{l} x=18p^4+2q^4-36p^2q^2+108p^3q-36pq^3, \\ y=-27p^4-3q^4+54p^2q^2+24p^3q-8pq^3 \end{array}\right\} \qquad (13)$$

Thus,(12) and (13) represent the integer solutions to (1).

Note 3

One may also write (1) in the form of ratios as

$$\frac{x+2z^2}{3z^2-y}=\frac{3(3z^2+y)}{x-2z^2}=\frac{\alpha}{\beta},\beta\neq 0,$$

$$\frac{x+2z^2}{3(3z^2-y)}=\frac{(3z^2+y)}{x-2z^2}=\frac{\alpha}{\beta},\beta\neq 0,$$

$$\frac{x+2z^2}{3(3z^2+y)}=\frac{(3z^2-y)}{x-2z^2}=\frac{\alpha}{\beta},\beta\neq 0$$

The repetition of the above process gives three more integer solutions to (1).

3. Conclusion

An attempt has been made to obtain non-zero distinct integer solutions to the non-homogeneous bi-quadratic diophantine equation with three unknowns given by $x^2+3y^2=31z^4$. One may search for other sets of integer solutions to the considered equation as well as other choices of the fourth degree diophantine equations with multi-variables

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