Extension of Ostrowski Inequality via the Katugampola Integrals of fractional Order

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Abstract

The purpose of proceeding work is to form new generalized and extended form of Ostrowski type inequality by the help of fraction order integral of Katugampola type which is the extended form of Reimann-Liouville and Hadamard integral of fractional order. To get the results, a new identity is introduced by the researchers which help in deriving the few inequalities for the class of whose powers are absolute derivatives values are p-convex. It can be seen that the extended form is generalization of some well-known results.

Keywords: p-covex functions, Reimann-Liouville, Katugampola Fractional Integral, Ostrowski Inequality

Introduction

Calculus of fractional order was acquainted with the start of the nineteenth century through the Liouville and Riemann. Related derivatives and other types of fractional local derivatives or modified conformable derivatives can gain in importance through their ability to generate more generalized non-local fractional derivatives with singular kernels [1-2]. Showed that several consequences in Mathematical programming touching on convex features in truth keep for a various elegance of features is known as Invex [3]. Introduced the idea of fuzzy preinvex features and invex sets. These features are more stylish than convex fuzzy features and while differentiable are fuzzy invex [4]. Studied on the deducement of Ostrowski Inequality considering Lipschitzan calculation and function tendency numerical search as well as considering Euler's beta inclined [5]. Established the advanced Ostrowski inequality and originates its conclusion to respect of the high and dip limits with derivative Also indicate a bit effort on Ostrowski inequality thoroughly continues function [6]. Extended analogous as well as particular versions for the documentation. He also applies their results to the segment calculation situation [7]. Enhanced various Ostrowski type inequality include restricted upon as well as double calculation associations [8]. Represented generalization of the Ostrowski integral inequality for mappings of restricted variation and requests for general quadrature formula [9]. Explained the basic definitions of invex set and η connected set and η -preinvex function η connected set with respect to bifunction and using this definition and from these results consider new class which is called logarithmically h -preinvex function also proved many inequalities for logarithmically h -preinvex by using fractional integral. Many special cases also investigated also explained that which results proved which can be extendable and helpful for further researchers [10]. Utilized settle interpretation of Ostrowski variation as long as aligning holder kind and cover analytical assimilation [11]. Studied a few sorts of invariant monotone plots and generalized invariant monotone plots are offered [12]. Enough optimality situations are received for a nonlinear a couple of objective fractional programming problem related to η -semi differentiable kind I-preinvex and related functions [13]. Represented a continuous function on time scales, an improved variation of Ostrowski's inequality is used [14]. Extended a time-scale variant for Ostrowski type inequality functions using limited derivatives and then extended the results in continuous and discrete cases [15]. Express companion of Ostrowski's integral inequality which applicable to a composite quadrature rule and to probability density function for differentiable mapping whose first derivative are bounded [16].

Used an important parameter that can find a generality of the acquaintance of Ostrowski type integral inequality whose second derivative relates to L^{∞} − Space which not only covers previous result but can also determines smaller estimator of the recorded results made by few conclusions of Ostrowski inequality use in numeric integration and attention on specific means was immediate [17]. Initiated several latest scaling variations to Ostrowski as well as Chebyshev functional for twice executes on period dimensions. Sustain testimonies would to distinct percentage as well as submit latest guesses on this kind variation [18]. Have presented the record of the major documents and events that have taken place in the field of fractional calculus sine 1974 up to the present date [19]. Established some significant Ostrowski's type inequities for m– and (α, m)-logarithm-mically convex purposes while utilizing the Riemann Liouville integral of fractional order [20]. Have derived some new type of Ostrowski inequality by using Riemann Liouville integral [21]. Have adjusted Ostrowski type inequity, trapezoid-type, Gruss type as well as Ostrowski-Gruss type inequity on proportionately [22]. Established Riemann–Liouville fractional integrals, novel generalizations of the Ostrowski inequality were discovered. There have also been certain specific cases explored [23]. Have established that n–polynomial exponential s–convex through fractional operator. A novel Hermite–Hadamard fractional integral inequity is also presented. There have also been several exceptional examples of the results presented.

In [24] Alexandar Markowich Ostrowski (1938) a Ukrainian Mathematician, identified an intriguing integral disparity known as the Ostrowski inequality. The absolute departure from the integral mean can be estimated using Ostrowski type inequalities. They might be utilized to calculate appropriate error boundaries for composite quadrature rules. This integral inequality has elegant and effective importance in numerical integration, optimization theory, integral operator theory, information, probability, statistics and special means. Ostrowski inequality might be interpret by utilizing Montgomery identity. Ostrowski initially presented his result for differential mapping, after that this result is generalized in many ways. It is massively crucial to get some fascinating inequalities, such as Ostrowski, due to the inevitable range of applications in many areas of mathematics; many mathematicians analyzed it at particular phases in different books and research papers. In numerical analysis, Ostrowski inequalities are quite crucial as they provide us the bounds of various quadrature's and also find its applications in probability theory and in special means.

Def. 1:[24] In 1938, Ostrowski devised the following fascinating integral inequality, which connects the value of a function f at one place in space to the value of a function f at another moment in time (ζ_1,ζ_2), and the integration on $[\zeta_1,\zeta_2]$

Let $f:[\zeta_1,\zeta_2]\rightarrow R$ be differentiable on (ζ_1,ζ_2), then we have

$$\left|f(r)-\frac{1}{\zeta_1-\zeta_2}\int_{\zeta_1}^{\zeta_2}f(s)ds\right|\le M(\zeta_1-\zeta_2)\left[\frac{\left(r-\frac{\zeta_1-\zeta_2}{2}\right)^2}{(\zeta_1-\zeta_2)^2}+\frac{1}{4}\right], \tag{1}$$

where $M=\sup_{\zeta_1<r<\zeta_2}\left|f^{'}(r)\right|<\infty$ holds for all $r\in[\zeta_1,\zeta_2]$. That's the Ostrowski Inequality, and the constant 1/4 is the finest solution.

Def. 2: [24] let $I\subset(0,\infty)$ be a real value interval and $p\in R\backslash\{0\}$. Then a function $f:I\rightarrow R$ is known as the *p*-covex function it follow the following one,

$$f\left(\left[\eta r^p+(1-\eta)\xi^p\right]^{\frac{1}{p}}\right)\le\eta f(r)+(1-\eta)f(\xi)\ \ \forall r,\xi\in I\ and\ \eta\in[0,1] \tag{2}$$

It can be reduced into ordinary convex function by putting $p=1$ and at $p=-1$ it behaves harmonically convex.

Def. 3: The integral Riemann-Liouville provides classical fractional method of calculus that is essentially. Weyl's integral is the theory of the periodic functions (having therefore the limited state of repeating after a certain time). It is set in the Fourier series and allows the constant Fourier coefficient to disappear (thus it applies to unit circle functions with integral valued 0). Let $[\lambda_1,\lambda_1](-\infty<\lambda_1<\lambda_2<\infty)$ be a limited interval on the real alliance ℝ. The Riemann Liouville integral of fractional order $J^{\alpha}_{\lambda_1^+}f$ and $J^{\alpha}_{\lambda_2^-}f$ of order $\alpha\in I(\Re(\alpha)>0)$ with $\lambda_1>0$ and $\lambda_2>0$ are defined respectively, by

$$J^{\alpha}_{\lambda_1^+}f=\frac{1}{\Gamma(\alpha)}\int_{\lambda_1}^{x}(x-t)^{\alpha-1}f(t)dt, \qquad (\lambda_1>a;\Re(\alpha)>0). \tag{3}$$

And

$$J_{\lambda_2^-}^{\alpha} f = \frac{1}{\Gamma(\alpha)} \int_x^{\lambda_2} (t-x)^{\alpha-1} f(t)\,dt, \qquad (\lambda_2 > b; \Re(\alpha) > 0). \tag{4}$$

where $\Gamma(t) = \int_0^{\infty} e^x x^{t-1} dx$ is a Euler Gamma Function [25].

Def. 4: [26] Let the space $\xi_c^t(\delta_1, \delta_2)(c \in R, 1 \le t \le \infty)$ of folks complex-valued measurable Lebesque functions f on $[\delta_1, \delta_2]$ for which $\| f \| \xi_c^t < \infty$, where the norm is defined by

$$\| f \| \xi_c^t = \left(\int_{\delta_1}^{\delta_2} \left| x^c f(x) \right|^t \frac{dx}{x} \right)^{\frac{1}{t}} < \infty \tag{5}$$

for $1 \le t \le \infty$, $c \in R$ and f for the case $t = \infty$,

$$\| f \| \xi_c^t = es\sup_{\delta_1 \le x \le \delta_2} \left[x^c \left| f(x) \right| \right], \qquad (c \in R).$$

Def. 5: [27] Katugampola has presented a new integral of fractional order which is generality of the Riemann-liouville and Hadmard integral of fractional order. For $[m,n] \subset R$, the statement for left and right sided the katugampola integral of fractional order for $\eta > 0$ is defined as

$$^{\mu}I_{m^+}^{\eta} k(l) = \frac{\mu^{1-\eta}}{\Gamma(\eta)} \int_m^l \frac{h^{\mu-1}}{(m^{\mu} - h^{\mu})^{1-\mu}} k(h)dh \tag{6}$$

$$^{\mu}I_{m^+}^{\eta} k(l) = \frac{\mu^{1-\eta}}{\Gamma(\eta)} \int_m^l \frac{h^{\mu-1}}{(m^{\mu} - h^{\mu})^{1-\mu}} k(h)dh \tag{7}$$

with $m < l < n$ and $\eta > 0$ if the intenal exists.

Def. 6: [27] The left and right side Hadamard integral of fractional order $\varphi \in R^+$ are demarcated as

$$\Im_{m^+}^{\varphi} \psi = \frac{1}{\Gamma(\varphi)} \int_m^h \frac{\psi(g)}{\left(\ln \frac{h}{g} \right)^{1-\psi}} \frac{dg}{g}, \quad h > m > 0,$$
$$\Im_{n^-}^{\varphi} \psi = \frac{1}{\Gamma(\varphi)} \int_h^n \frac{\psi(g)}{\left(\ln \frac{g}{h} \right)^{1-\psi}} \frac{dg}{g}, \quad 0 < h < n, \tag{8}$$

Results

Extension of Ostrowski inequality for Katugampola fractional integral

In this paper, applying the Katugampola fractional identity, we established the extensions of Ostrowski type inequalities for Katugampola fractional integrals. The notation $D_k(\eta, \mu; m, l, n)$ will be used in lemmas and theorems as;

$$D_k(\eta,\mu;m,l,n)=\frac{\mu k(l)}{n-m}\left[\left(l^{\mu}-m^{\mu}\right)^{\mu}+\left(n^{\mu}-l^{\mu}\right)^{\mu}\right]-\frac{\mu^{\eta+1}\Gamma(\eta+1)}{n-m}\left[{}^{\mu}\mathrm{I}^{\eta}_{l^-}k(m)+{}^{\mu}\mathrm{I}^{\eta}_{l^+}k(n)\right] \tag{9}$$

Theorem 1

Let $k:I\subset(0,\infty)\to R$ be a differentiable mapping on I^{o} where $m,n\in I$ with $m<n$. If $k^{'}\in L[m,n]$, then $\forall l\in[m,n]$ the following identity holds

$$D_k(\eta,\mu;m,l,n)=\frac{\left(l^{\mu}-m^{\mu}\right)^{\eta}}{n-m}\int_0^1\frac{r^{\eta}k^{'}\left(\left[rl^{\mu}+(1-r)m^{\mu}\right]^{\frac{1}{\mu}}\right)}{\left[rl^{\mu}+(1-r)m^{\mu}\right]^{1-\frac{1}{\mu}}}dr-\frac{\left(n^{\mu}-l^{\mu}\right)^{\eta+1}}{n-m}\int_0^1\frac{r^{\eta}k^{'}\left(\left[rl^{\mu}+(1-r)n^{\mu}\right]^{\frac{1}{\mu}}\right)}{\left[rl^{\mu}+(1-r)n^{\mu}\right]^{1-\frac{1}{\mu}}}dr \tag{10}$$

Where $\eta>0,\mu>0$.

Proof

Let the integral of equation (10) be,

$$I_1=\int_0^1\frac{r^{\eta}k^{'}\left(\left[rl^{\mu}+(1-r)m^{\mu}\right]^{\frac{1}{\mu}}\right)}{\left[rl^{\mu}+(1-r)m^{\mu}\right]^{1-\frac{1}{\mu}}}dr, \tag{11}$$

And

$$I_2=\int_0^1\frac{r^{\eta}k^{'}\left(\left[rl^{\mu}+(1-r)n^{\mu}\right]^{\frac{1}{\mu}}\right)}{\left[rl^{\mu}+(1-r)n^{\mu}\right]^{1-\frac{1}{\mu}}}dr, \tag{12}$$

Integrating by parts, equation (11) becomes

$$I_1=r^{\eta}\frac{\mu k\left(rl^{\mu}+(1-r)m^{\mu}\right)^{\frac{1}{\mu}}}{l^{\mu}-m^{\mu}}\Bigg|_0^1-\int_0^1\eta r^{\eta-1}\frac{\mu k\left(rl^{\mu}+(1-r)m^{\mu}\right)^{\frac{1}{\mu}}}{l^{\mu}-m^{\mu}}dr$$

$$=\frac{\mu}{l^{\mu}-m^{\mu}}\left[1.k\left(l^{\mu}\right)^{\frac{1}{\mu}}-0\right]-\frac{\eta\mu}{l^{\mu}-m^{\mu}}\int_0^1 r^{\eta-1}\mu k\left(rl^{\mu}+(1-r)m^{\mu}\right)^{\frac{1}{\mu}}dr$$

$$=\frac{\mu k(l)}{l^{\mu}-m^{\mu}}-\frac{\eta\mu}{l^{\mu}-m^{\mu}}\int_0^1 r^{\eta-1}\mu k\left(rl^{\mu}+(1-r)m^{\mu}\right)^{\frac{1}{\mu}}dr, \tag{13}$$

put $w=\left(rl^{\mu}+(1-r)m^{\mu}\right)^{\frac{1}{\mu}}$ in equation (13),

$$w^{\mu} = rl^{\mu} + (1-r)m^{\mu}$$
$$w^{\mu} = rl^{\mu} + m^{\mu} - rm^{\mu}$$
$$w^{\mu} - m^{\mu} = r\left(l^{\mu} - m^{\mu}\right)$$
$$r = \frac{w^{\mu} - m^{\mu}}{\left(l^{\mu} - m^{\mu}\right)}$$
$$\frac{\mu w^{\mu-1} dw}{l^{\mu} - m^{\mu}} = dr$$
$$when;$$
$$t = 1, w = l$$
$$t = 0, w = m$$

Then, equation (13) becomes

$$I_1 = \frac{\mu}{l^{\mu} - m^{\mu}} - \frac{\eta\mu}{l^{\mu} - m^{\mu}} \int_m^l \left(\frac{w^{\mu} - m^{\mu}}{l^{\mu} - m^{\mu}} \right)^{\eta-1} k(w) \frac{\mu w^{\eta-1}}{l^{\mu} - m^{\mu}} dw,$$
$$= \frac{\mu}{l^{\mu} - m^{\mu}} - \frac{\eta\mu^2}{\left(l^{\mu} - m^{\mu}\right)^{\eta+1}} \int_m^l \frac{w^{\eta-1}}{l^{\mu} - m^{\mu}} k(w) dw, \tag{14}$$

Multiplying and dividing equation (14) with $\mu^{1-\eta}\Gamma(\eta)$,

$$I_1 = \frac{\mu k(l)}{l^{\mu} - m^{\mu}} - \frac{\eta\mu^2}{\left(l^{\mu} - m^{\mu}\right)^{\eta+1}} \frac{\mu^{1-\eta}\Gamma(\eta)}{\mu^{1-\eta}\Gamma(\eta)} \int_m^l \frac{w^{\eta-1}}{l^{\mu} - m^{\mu}} k(w) dw$$
$$= \frac{\mu k(l)}{l^{\mu} - m^{\mu}} - \frac{\eta\mu^2\left(\Gamma(\eta)\right)}{\left(l^{\mu} - m^{\mu}\right)^{\eta+1}\left(\mu^{1-\eta}\right)} \left(\frac{\mu^{1-\eta}}{\Gamma(\eta)} \int_m^l \frac{w^{\eta-1}}{l^{\mu} - m^{\mu}} k(w) dw \right)$$
$$I_1 = \frac{\mu k(l)}{l^{\mu} - m^{\mu}} - \frac{\mu^2\left(\eta\Gamma(\eta)\right)}{\left(l^{\mu} - m^{\mu}\right)^{\eta+1}\left(\mu^{1-\eta}\right)} \left(\frac{\mu^{1-\eta}}{\Gamma(\eta)} \int_m^l \frac{w^{\eta-1}}{l^{\mu} - m^{\mu}} k(w) dw \right), \tag{15}$$

Then by definition (5), equation (15) become

$$I_1 = \frac{\mu k(l)}{l^{\mu} - m^{\mu}} - \frac{\eta\mu^2\left(\Gamma(\eta+1)\right)}{\left(l^{\mu} - m^{\mu}\right)^{\eta+1}\left(\mu^{1-\eta}\right)} \, {}^{\mu}I_{l^-}^{\eta} k(m) \tag{16}$$

Similarly, by changing the variable ‘m’ into ‘n’ in equation (16), equation (12) becomes,

$$I_2 = \frac{\mu k(l)}{l^{\mu} - m^{\mu}} - \frac{\mu^2\left(\eta\Gamma(\eta)\right)}{\left(l^{\mu} - m^{\mu}\right)^{\eta+1}\left(\mu^{1-\eta}\right)} \left(\frac{\mu^{1-\eta}}{\Gamma(\eta)} \int_n^l \frac{w^{\eta-1}}{l^{\mu} - n^{\mu}} k(w) dw \right)$$
$$= -\frac{\mu k(l)}{n^{\mu} - l^{\mu}} + \frac{\mu^2\left(\Gamma(\eta+1)\right)}{\left(n^{\mu} - l^{\mu}\right)^{\eta+1}\left(\mu^{1-\eta}\right)} \left(\frac{\mu^{1-\eta}}{\Gamma(\eta)} \int_l^n \frac{w^{\eta-1}}{n^{\mu} - l^{\mu}} k(w) dw \right) \tag{17}$$

Then by definition (6), equation (17) becomes

$$I_2 = -\frac{\mu k(l)}{n^\mu - l^\mu} + \frac{\mu^2 (\Gamma(\eta+1))}{(n^\mu - l^\mu)^{\eta+1} (\mu^{1-\eta})} {}^{\mu}I_{l^+}^{\eta} k(n) \tag{18}$$

Multiplying equation (16) with $\frac{(l^\mu - m^\mu)^{\eta+1}}{n-m}$ and equation (18) with $-\frac{(n^\mu - l^\mu)^{\eta+1}}{n-m}$ and then by adding both the equations, it becomes,

$$\begin{aligned}
\frac{(l^\mu - m^\mu)^{\eta+1}}{n-m} I_1 + \left(-\frac{(n^\mu - l^\mu)^{\eta+1}}{n-m}\right) I_2 &= \frac{(l^\mu - m^\mu)^{\eta+1}}{n-m}\left(\frac{\mu k(l)}{l^\mu - m^\mu} - \frac{\eta\mu^2(\Gamma(\eta+1))}{(l^\mu - m^\mu)^{\eta+1}(\mu^{1-\eta})} {}^{\mu}I_{l^-}^{\eta} k(m)\right) + \\
&\left(-\frac{(n^\mu - l^\mu)^{\eta+1}}{n-m}\right)\left(-\frac{\mu k(l)}{n^\mu - l^\mu} + \frac{\mu^2(\Gamma(\eta+1))}{(n^\mu - l^\mu)^{\eta+1}(\mu^{1-\eta})} {}^{\mu}I_{l^+}^{\eta} k(n)\right) \\
&= \frac{\mu k(l)}{n-m}\left[(l^\mu - m^\mu)^\mu + (n^\mu - l^\mu)^\mu\right] \\
&\quad - \frac{\mu^{\eta+1}\Gamma(\eta+1)}{n-m}\left[{}^{\mu}\mathrm{I}_{l^-}^{\eta} k(m) + {}^{\mu}\mathrm{I}_{l^+}^{\eta} k(n)\right]
\end{aligned} \tag{19}$$

Then from equation (9), equation (19) becomes,

$$\frac{(l^\mu - m^\mu)^{\eta+1}}{n-m} I_1 + \left(-\frac{(n^\mu - l^\mu)^{\eta+1}}{n-m}\right) I_2 = D_k(\eta, \mu; m, l, n) \tag{20}$$

From equation (11) and (12) equation (20) becomes

$$\begin{aligned}
D_k(\eta, \mu; m, l, n) &= \frac{(l^\mu - m^\mu)^\eta}{n-m} \int_0^1 \frac{r^\eta k'\left(\left[rl^\mu + (1-r)m^\mu\right]^{\frac{1}{\mu}}\right)}{\left[rl^\mu + (1-r)m^\mu\right]^{1-\frac{1}{\mu}}} dr \\
&\quad - \frac{(n^\mu - l^\mu)^{\eta+1}}{n-m} \int_0^1 \frac{r^\eta k'\left(\left[rl^\mu + (1-r)n^\mu\right]^{\frac{1}{\mu}}\right)}{\left[rl^\mu + (1-r)n^\mu\right]^{1-\frac{1}{\mu}}} dr.
\end{aligned} \tag{21}$$

Theorem 2

Let $k : I \subset (0,\infty) \to R$ be a differentiable mapping on I^o where $m, n \in I$ with $m < n$ such that $k' \in L[m,n]$. If $|k'|$ is p-convex on I and $|k'(l)| \le M \ \forall l \in \left[m, 2^{\frac{1}{\mu}} m\right]\left(if\ 2^{\frac{1}{\mu}} m < n, otherwise, l \in [m,n]\right)$, then $\forall l \in [m,n]$ the following identity holds

$$|D_k(\eta, \mu; m, l, n)| \le M \frac{(l^\mu - m^\mu)^{\eta+1}}{n-m}\{R(m) + S(m)\} + M \frac{(n^\mu - l^\mu)^{\eta+1}}{n-m}\{R(n) + S(n)\}$$

Where,

$$R(\xi) = \frac{\xi^{1-\mu}}{\eta+2} {}_2F_1\left(\eta+2, \frac{\mu-1}{\mu}; \eta+3; 1 - \frac{l^\mu}{\xi^\mu}\right) \tag{22}$$

$$S(\xi)=\frac{\xi^{1-\eta}}{(\eta+1)(\eta+2)}\left[\begin{aligned}&(\eta+2)\,{}_2F_1\left(\eta+1,\frac{\mu-1}{\mu};\eta+2;1-\frac{l^{\mu}}{\xi^{\mu}}\right)\\&-(\eta+1)\,{}_2F_1\left(\eta+2,\frac{\mu-1}{\mu};\eta+3;1-\frac{l^{\mu}}{\xi^{\mu}}\right)\end{aligned}\right] \tag{23}$$

And $\mu>1,\eta>0,\xi\in\{m,n\}$, ${}_2F_1(.,.;.;.)$ is a hypergeometric function.

Proof

By using lemma and properties of modulus, it can be written as,

$$\begin{aligned}\left|D_k(\eta,\mu;m,l,n)\right|\le&\frac{(l^{\mu}-m^{\mu})^{\eta}}{n-m}\int_0^1\frac{r^{\eta}\left|k'\left(\left[rl^{\mu}+(1-r)m^{\mu}\right]^{\frac{1}{\mu}}\right)\right|}{\left[rl^{\mu}+(1-r)m^{\mu}\right]^{1-\frac{1}{\mu}}}dr\\&+\frac{(n^{\mu}-l^{\mu})^{\eta+1}}{n-m}\int_0^1\frac{r^{\eta}\left|k'\left(\left[rl^{\mu}+(1-r)n^{\mu}\right]^{\frac{1}{\mu}}\right)\right|}{\left[rl^{\mu}+(1-r)n^{\mu}\right]^{1-\frac{1}{\mu}}}dr\end{aligned} \tag{24}$$

By p-convexity of $|k'|$, from equation (24) following computation can be formed,

$$\begin{aligned}\left|D_k(\eta,\mu;m,l,n)\right|\le&\frac{(l^{\mu}-m^{\mu})^{\eta}}{n-m}\int_0^1\frac{r^{\eta}\left[r|k'(l)|+(1-r)|k'(m)|\right]}{\left[rl^{\mu}+(1-r)m^{\mu}\right]^{1-\frac{1}{\mu}}}dr\\&+\frac{(n^{\mu}-l^{\mu})^{\eta+1}}{n-m}\int_0^1\frac{r^{\eta}\left[r|k'(l)|+(1-r)|k'(n)|\right]}{\left[rl^{\mu}+(1-r)n^{\mu}\right]^{1-\frac{1}{\mu}}}dr\\\le&\frac{(l^{\mu}-m^{\mu})^{\eta}}{n-m}\left(\int_0^1\frac{r^{\eta}\left[r|k'(l)|\right]}{\left[rl^{\mu}+(1-r)m^{\mu}\right]^{1-\frac{1}{\mu}}}dr+\int_0^1\frac{r^{\eta}(1-r)|k'(m)|}{\left[rl^{\mu}+(1-r)m^{\mu}\right]^{1-\frac{1}{\mu}}}dr\right)\\&+\frac{(n^{\mu}-l^{\mu})^{\eta+1}}{n-m}\left(\int_0^1\frac{r^{\eta}\left[r|k'(l)|\right]}{\left[rl^{\mu}+(1-r)n^{\mu}\right]^{1-\frac{1}{\mu}}}dr+\int_0^1\frac{r^{\eta}(1-r)|k'(n)|}{\left[rl^{\mu}+(1-r)n^{\mu}\right]^{1-\frac{1}{\mu}}}dr\right)\\\le&\frac{(l^{\mu}-m^{\mu})^{\eta}}{n-m}\Bigg[|k'(l)|\int_0^1 r^{\eta+1}\left[rl^{\mu}+(1-r)m^{\mu}\right]^{\frac{1}{\mu}-1}dr\\&\qquad+|k'(m)|\int_0^1\left(r^{\eta}-r^{\eta+1}\right)\left[rl^{\mu}+(1-r)m^{\mu}\right]^{\frac{1}{\mu}-1}dr\Bigg]\\&+\frac{(n^{\mu}-l^{\mu})^{\eta+1}}{n-m}\Bigg[|k'(l)|\int_0^1 r^{\eta+1}\left[rl^{\mu}+(1-r)n^{\mu}\right]^{\frac{1}{\mu}-1}dr\\&\qquad+|k'(n)|\int_0^1\left(r^{\eta}-r^{\eta+1}\right)\left[rl^{\mu}+(1-r)n^{\mu}\right]^{\frac{1}{\mu}-1}dr\Bigg]\end{aligned} \tag{25}$$

Let the integrals from equation (25) be,

$$I_1 = \int_0^1 r^{\eta+1}\left[rl^{\mu} + (1-r)m^{\mu}\right]^{\frac{1}{\mu}-1} dr$$

$$I_2 = \int_0^1 \left(r^{\eta} - r^{\eta+1}\right)\left[rl^{\mu} + (1-r)m^{\mu}\right]^{\frac{1}{\mu}-1} dr$$

$$I_3 = \int_0^1 r^{\eta+1}\left[rl^{\mu} + (1-r)n^{\mu}\right]^{\frac{1}{\mu}-1} dr$$

$$I_4 = \int_0^1 \left(r^{\eta} - r^{\eta+1}\right)\left[rl^{\mu} + (1-r)n^{\mu}\right]^{\frac{1}{\mu}-1} dr$$

Then solving the integral I_1 from equation (25),

$$\begin{aligned} I_1 &= \int_0^1 r^{\eta+1}\left[rl^{\mu} + (1-r)m^{\mu}\right]^{\frac{1}{\mu}-1} dr \\ &= \int_0^1 r^{\eta+1}\left[rl^{\mu} + m^{\mu} - rm^{\mu}\right]^{\frac{1}{\mu}-1} dr \\ &= \left(m^{\mu}\right)^{\frac{1}{\mu}-1}\int_0^1 r^{\eta+1}\left[r\frac{l^{\mu}}{m^{\mu}} + 1 - r\right]^{\frac{1}{\mu}-1} dr \\ &= \left(m^{\mu}\right)^{\frac{1}{\mu}-1}\int_0^1 r^{\eta+1}\left[1 - r\left(1 - \frac{l^{\mu}}{m^{\mu}}\right)\right]^{\frac{1}{\mu}-1} dr \\ &= \left(m^{\mu}\right)^{\frac{1}{\mu}-1}\int_0^1 r^{\eta+2-1}(1-r)^{\eta+3-(\eta+2)-1}\left[1 - r\left(1 - \frac{l^{\mu}}{m^{\mu}}\right)\right]^{-\left(1-\frac{1}{\mu}\right)} dr \\ &= \left(m^{\mu}\right)^{\frac{1}{\mu}-1}\int_0^1 r^{\eta+2-1}(1-r)^{\eta+3-(\eta+2)-1}\left[1 - r\left(1 - \frac{l^{\mu}}{m^{\mu}}\right)\right]^{-\left(\frac{\mu-1}{\mu}\right)} dr, \end{aligned} \tag{26}$$

Multiplying and dividing equation (26) by $\beta(\eta+2,1)$,

$$\begin{aligned} I_1 &= \left(m^{\mu}\right)^{\frac{1}{\mu}-1}\frac{\beta(\eta+2,1)}{\beta(\eta+2,1)}\int_0^1 r^{\eta+2-1}(1-r)^{\eta+3-(\eta+2)-1}\left[1 - r\left(1 - \frac{l^{\mu}}{m^{\mu}}\right)\right]^{-\left(\frac{\mu-1}{\mu}\right)} dr \\ &= \left(m^{\mu}\right)^{\frac{1}{\mu}-1}\beta(\eta+2,1)\left(\frac{1}{\beta(\eta+2,1)}\int_0^1 r^{\eta+2-1}(1-r)^{\eta+3-(\eta+2)-1}\left[1 - r\left(1 - \frac{l^{\mu}}{m^{\mu}}\right)\right]^{-\left(\frac{\mu-1}{\mu}\right)} dr\right) \\ &= \left(m^{\mu}\right)^{\frac{1}{\mu}-1}\beta(\eta+2,1)\left(\frac{1}{\beta(\eta+2,\eta+3-(\eta+2))}\int_0^1 r^{\eta+2-1}(1-r)^{\eta+3-(\eta+2)-1}\left[1 - r\left(1 - \frac{l^{\mu}}{m^{\mu}}\right)\right]^{-\left(\frac{\mu-1}{\mu}\right)} dr\right) \end{aligned} \tag{27}$$

Then by definition of hypper geometric function, equation (27) becomes,

$$I_1 = \left(m^\mu\right)^{\frac{1}{\mu}-1} \beta(\eta+2,1)\, {}_2F_1\left(\eta+2, \frac{\mu-1}{\mu}; \eta+3; 1-\frac{l^\mu}{m^\mu}\right), \tag{28}$$

By expanding beta function into gamma function, equation (28) becomes

$$\begin{aligned} I_1 &= \left(m^\mu\right)^{\frac{1}{\mu}-1} \left(\frac{\Gamma(\eta+2)\Gamma(1)}{\Gamma(\eta+2+1)}\right) {}_2F_1\left(\eta+2, \frac{\mu-1}{\mu}; \eta+3; 1-\frac{l^\mu}{m^\mu}\right) \\ &= \left(m^\mu\right)^{\frac{1}{\mu}-1} \left(\frac{\Gamma(\eta+2)\Gamma(1)}{\Gamma(\eta+3)}\right) {}_2F_1\left(\eta+2, \frac{\mu-1}{\mu}; \eta+3; 1-\frac{l^\mu}{m^\mu}\right) \\ &= \left(m^\mu\right)^{\frac{1}{\mu}-1} \left(\frac{\Gamma(\eta+2)}{(\eta+2)\Gamma(\eta+2)}\right) {}_2F_1\left(\eta+2, \frac{\mu-1}{\mu}; \eta+3; 1-\frac{l^\mu}{m^\mu}\right) \\ &= \frac{\left(m^\mu\right)^{\frac{1}{\mu}-1}}{(\eta+2)}\, {}_2F_1\left(\eta+2, \frac{\mu-1}{\mu}; \eta+3; 1-\frac{l^\mu}{m^\mu}\right), \end{aligned} \tag{29}$$

Then by equation (22), equation (29) becomes,

$$I_1 = R(m) \tag{30}$$

Now solving the integrals I_2, from equation (25)

$$\begin{aligned} I_2 &= \int_0^1 \left(r^\eta - r^{\eta+1}\right)\left[rl^\mu + (1-r)m^\mu\right]^{\frac{1}{\mu}-1} dr \\ &= \int_0^1 \left(r^\eta - r^{\eta+1}\right)\left[rl^\mu + m^\mu - rm^\mu\right]^{\frac{1}{\mu}-1} dr \\ &= \left(m^\mu\right)^{\frac{1}{\mu}-1} \int_0^1 \left(r^\eta - r^{\eta+1}\right)\left[r\frac{l^\mu}{m^\mu} + 1 - r\right]^{\frac{1}{\mu}-1} dr \\ &= \left(m^\mu\right)^{\frac{1}{\mu}-1} \left(\int_0^1 r^\eta\left[1 - r\left(1-\frac{l^\mu}{m^\mu}\right)\right]^{\frac{1}{\mu}-1} dr - \int_0^1 r^\eta\left[1 - r\left(1-\frac{l^\mu}{m^\mu}\right)\right]^{\frac{1}{\mu}-1} dr\right) \\ &= \left(m^\mu\right)^{\frac{1}{\mu}-1} \left[\int_0^1 r^{\eta+1-1}(1-r)^{\eta+2-(\eta+1)-1}\left[1 - r\left(1-\frac{l^\mu}{m^\mu}\right)\right]^{-\left(1-\frac{1}{\mu}\right)} dr \right. \\ &\qquad \left. - \int_0^1 r^{\eta+2-1}(1-r)^{\eta+3-(\eta+2)-1}\left[1 - r\left(1-\frac{l^\mu}{m^\mu}\right)\right]^{-\left(1-\frac{1}{\mu}\right)} dr\right], \end{aligned} \tag{31}$$

Multiplying and dividing equation (31) by $\beta(\eta+2,1)$ and $\beta(\eta+1,1)$,

$$I_2=\left(m^{\mu}\right)^{\frac{1}{\mu}-1}\frac{\beta(\eta+2,1).\beta(\eta+1,1)}{\beta(\eta+2,1).\beta(\eta+1,1)}\left[\int_0^1 r^{\eta+1-1}(1-r)^{\eta+2-(\eta+1)-1}\left[1-r\left(1-\frac{l^{\mu}}{m^{\mu}}\right)\right]^{-\left(1-\frac{1}{\mu}\right)}dr\right.$$

$$\left.-\int_0^1 r^{\eta+2-1}(1-r)^{\eta+3-(\eta+2)-1}\left[1-r\left(1-\frac{l^{\mu}}{m^{\mu}}\right)\right]^{-\left(1-\frac{1}{\mu}\right)}dr\right]$$

$$=\left(m^{\mu}\right)^{\frac{1}{\mu}-1}\beta(\eta+2,1).\beta(\eta+1,1)\left[\frac{1}{\beta(\eta+2,1)}\left(\frac{1}{\beta(\eta+1,1)}\int_0^1 r^{\eta+1-1}(1-r)^{\eta+2-(\eta+1)-1}\left[1-r\left(1-\frac{l^{\mu}}{m^{\mu}}\right)\right]^{-\left(1-\frac{1}{\mu}\right)}dr\right)\right.$$

$$\left.-\frac{1}{\beta(\eta+1,1)}\left(\frac{1}{\beta(\eta+2,1)}\int_0^1 r^{\eta+2-1}(1-r)^{\eta+3-(\eta+2)-1}\left[1-r\left(1-\frac{l^{\mu}}{m^{\mu}}\right)\right]^{-\left(1-\frac{1}{\mu}\right)}dr\right)\right], \tag{32}$$

Then by definition of Hypper-Geometric function, equation (32) becomes,

$$I_2=\left(m^{\mu}\right)^{\frac{1}{\mu}-1}\beta(\eta+2,1).\beta(\eta+1,1)\left[\frac{1}{\beta(\eta+2,1)}\,{}_2F_1\left(\eta+1,\frac{\mu-1}{\mu};\eta+2;1-\frac{l^{\mu}}{m^{\mu}}\right)\right.$$

$$\left.-\frac{1}{\beta(\eta+1,1)}\,{}_2F_1\left(\eta+2,\frac{\mu-1}{\mu};\eta+3;1-\frac{l^{\mu}}{m^{\mu}}\right)\right]$$

$$=\left(m^{\mu}\right)^{\frac{1}{\mu}-1}\frac{\Gamma(\eta+2)\Gamma(1)}{\Gamma(\eta+2+1)}.\frac{\Gamma(\eta+1)\Gamma(1)}{\Gamma(\eta+1+1)}\left[\frac{\Gamma(\eta+2+1)}{\Gamma(\eta+2)\Gamma(1)}\,{}_2F_1\left(\eta+1,\frac{\mu-1}{\mu};\eta+2;1-\frac{l^{\mu}}{m^{\mu}}\right)\right.$$

$$\left.-\frac{\Gamma(\eta+1+1)}{\Gamma(\eta+1)\Gamma(1)}\,{}_2F_1\left(\eta+2,\frac{\mu-1}{\mu};\eta+3;1-\frac{l^{\mu}}{m^{\mu}}\right)\right]$$

$$=\left(m^{\mu}\right)^{\frac{1}{\mu}-1}\frac{1}{(\eta+2)}.\frac{1}{(\eta+1)}\left[(\eta+2)\,{}_2F_1\left(\eta+1,\frac{\mu-1}{\mu};\eta+2;1-\frac{l^{\mu}}{m^{\mu}}\right)\right.$$

$$\left.-(\eta+1)\,{}_2F_1\left(\eta+2,\frac{\mu-1}{\mu};\eta+3;1-\frac{l^{\mu}}{m^{\mu}}\right)\right], \tag{33}$$

Then by equation (23), equation (33) becomes,

$$I_2=S(m). \tag{34}$$

For integral of equation (25) I_3 and I_4, changing the variable 'm' into 'n' in equation (29) and (33), it becomes,

$$I_3=\frac{\left(n^{\mu}\right)^{\frac{1}{\mu}-1}}{(\eta+2)}\,{}_2F_1\left(\eta+2,\frac{\mu-1}{\mu};\eta+3;1-\frac{l^{\mu}}{n^{\mu}}\right), \tag{35}$$

$$I_4=\left(n^{\mu}\right)^{\frac{1}{\mu}-1}\frac{1}{(\eta+2)}.\frac{1}{(\eta+1)}\left[(\eta+1)\,{}_2F_1\left(\eta+2,\frac{\mu-1}{\mu};\eta+3;1-\frac{l^{\mu}}{n^{\mu}}\right)\right.$$

$$\left.-(\eta+2)\,{}_2F_1\left(\eta+1,\frac{\mu-1}{\mu};\eta+2;1-\frac{l^{\mu}}{n^{\mu}}\right)\right], \tag{36}$$

Then from equation (32) and (33), equation (35) and (36) can be written as

$$I_3 = R(n). \tag{37}$$

And

$$I_4 = S(n). \tag{38}$$

Then from equation (30), (34), (37) and (38), equation (35) becomes,

$$\begin{aligned} |D_k(\eta,\mu;m,l,n)| \le \frac{(l^\mu - m^\mu)^\eta}{n-m}\left(|k'(l)|R(m) + |k'(m)|S(m)\right) \\ + \frac{(n^\mu - l^\mu)^{\eta+1}}{n-m}\left(|k'(l)|R(n) + |k'(n)|S(n)\right), \end{aligned} \tag{39}$$

by boundedness of $k'(l)$, equation (39) becomes,

$$|D_k(\eta,\mu;m,l,n)| \le M\frac{(l^\mu - m^\mu)^{\eta+1}}{n-m}\{R(m)+S(m)\} + M\frac{(n^\mu - l^\mu)^{\eta+1}}{n-m}\{R(n)+S(n)\}$$

Theorem 3

Let $k : I \subset (0,\infty) \to R$ be a differentiable mapping on I^o where $m,n \in I$ with $m<n$ such that $k' \in L[m,n]$. If $|k'|^q$ is p-convex on I and $|k'(l)| \le M \ \forall l \in I - \{m,n\}$, then the following identity holds

$$|D_k(\eta,\mu;m,l,n)| \le \frac{M}{n-m}\left(\frac{1}{\eta q+1}\right)^{\frac{1}{q}}\left[(l^\mu - m^\mu)^{\eta+1}P^{\frac{1}{r}}(m) + (n^\mu - l^\mu)^{\eta+1}P^{\frac{1}{r}}(n)\right],$$

Where,

$$P(\xi) = \frac{\eta\left(l^{r(1-\mu)+\mu} - \xi^{r(1-\mu)+\mu}\right)}{(l^\mu - \xi^\mu)(r(1-\mu)+\mu)}, \tag{40}$$

and $\mu > 1, \eta > 0, \xi \in \{m,n\}, r > 1, \frac{1}{r} + \frac{1}{q} = 1, r \ne \frac{\eta}{\eta - 1}$.

Proof

By using lemma and properties of modulus, it can be written as,

$$\begin{aligned} |D_k(\eta,\mu;m,l,n)| \le \frac{(l^\mu - m^\mu)^\eta}{n-m}\int_0^1 \frac{h^\eta \left|k'\left(\left[hl^\mu + (1-h)m^\mu\right]^{\frac{1}{\mu}}\right)\right|}{\left[hl^\mu + (1-h)m^\mu\right]^{1-\frac{1}{\mu}}}dh \\ + \frac{(n^\mu - l^\mu)^{\eta+1}}{n-m}\int_0^1 \frac{h^\eta \left|k'\left(\left[hl^\mu + (1-h)n^\mu\right]^{\frac{1}{\mu}}\right)\right|}{\left[hl^\mu + (1-h)n^\mu\right]^{1-\frac{1}{\mu}}}dh, \end{aligned} \tag{41}$$

by using Holder inequality, equation (41) can be written as,

$$\left|D_k(\eta,\mu;m,l,n)\right| \le \frac{\left(l^\mu-m^\mu\right)^\eta}{n-m}\left[\left(\int_0^1\left(\left(hl^\mu+(1-h)m^\mu\right)^{\frac{1}{\mu}-1}\right)^r dh\right)^{\frac{1}{r}}\right.$$

$$\left.\left(\int_0^1 h^\eta\left|k'\left(\left[hl^\mu+(1-h)m^\mu\right]^{\frac{1}{\mu}}\right)\right|^q dh\right)^{\frac{1}{q}}\right]$$

$$+\frac{\left(n^\mu-l^\mu\right)^{\eta+1}}{n-m}\left[\left(\int_0^1\left(\left(hl^\mu+(1-h)n^\mu\right)^{\frac{1}{\mu}-1}\right)^r dh\right)^{\frac{1}{r}}\right.$$

$$\left.\left(\int_0^1 h^\eta\left|k'\left(\left[hl^\mu+(1-h)n^\mu\right]^{\frac{1}{\mu}}\right)\right|^q dh\right)^{\frac{1}{q}}\right] \tag{42}$$

Let the integrals of equation (42) be,

$$I_1=\int_0^1\left(\left(hl^\mu+(1-h)m^\mu\right)^{\frac{1}{\mu}-1}\right)^r dh. \tag{43}$$

$$I_2=\int_0^1\left(\left(hl^\mu+(1-h)n^\mu\right)^{\frac{1}{\mu}-1}\right)^r dh. \tag{44}$$

Then, put $w=hl^\mu+(1-h)m^\mu$ in equation (43),

$$w=hl^\mu+m^\mu-hm^\mu$$

$$w=hl^\mu+m^\mu-hm^\mu$$

$$w-m^\mu=h\left(l^\mu-m^\mu\right),$$

$$h=\frac{w-m^\mu}{\left(l^\mu-m^\mu\right)}$$

$$\frac{dw}{l^\mu-m^\mu}=dh$$

When,

$$t=1, w=l^\mu$$

$$t=0, w=m^\mu$$

$$I_1 = \frac{1}{l^\mu - m^\mu} \int_{m^\mu}^{l^\mu} (w)^{\left(\frac{1}{\mu}-1\right)r}$$

$$= \frac{1}{l^\mu - m^\mu} \left. \frac{(w)^{\left(\frac{1}{\mu}-1\right)r+1}}{\left(\left(\frac{1}{\mu}-1\right)r+1\right)} \right|_{m^\mu}^{l^\mu}$$

$$= \frac{1}{\left(l^\mu - m^\mu\right)\left(\left(\frac{1-\mu}{\mu}\right)r+1\right)} \left[\left(l^\mu\right)^{\left(\frac{1-\mu}{\mu}\right)r+1} - \left(m^\mu\right)^{\left(\frac{1-\mu}{\mu}\right)r+1} \right]$$

$$= \frac{\mu}{\left(l^\mu - m^\mu\right)\left((1-\mu)r+\mu\right)} \left[\left(l^\mu\right)^{\left(\frac{(1-\mu)r+\mu}{\mu}\right)} - \left(m^\mu\right)^{\left(\frac{(1-\mu)r+\mu}{\mu}\right)} \right]$$

$$= \frac{\mu\left[l^{(1-\mu)r+\mu} + m^{(1-\mu)r+\mu} \right]}{\left(l^\mu - m^\mu\right)\left((1-\mu)r+\mu\right)}, \tag{45}$$

Then from equation (40), equation (45) becomes

$$I_1 = P(m). \tag{46}$$

For integral I_2 of equation (44), changing the variable 'm' into 'n' in equation (45), it becomes

$$I_2 = \frac{\mu\left[l^{(1-\mu)r+\mu} + n^{(1-\mu)r+\mu} \right]}{\left(l^\mu - n^\mu\right)\left((1-\mu)r+\mu\right)}, \tag{47}$$

Then from equation (40), equation (47) becomes,

$$I_2 = P(n). \tag{48}$$

Then from equation (46) and (48), equation (42) becomes

$$\left|D_k(\eta,\mu;m,l,n)\right| \le \frac{\left(l^\mu - m^\mu\right)^\eta}{n-m} \left[\left(P(m)\right)^{\frac{1}{r}} \left(\int_0^1 h^\eta \left| k'\left(\left[hl^\mu + (1-h)m^\mu \right]^{\frac{1}{\mu}} \right) \right|^q dh \right)^{\frac{1}{q}} \right] + \frac{\left(n^\mu - l^\mu\right)^{\eta+1}}{n-m} \left[\left(P(n)\right)^{\frac{1}{r}} \left(\int_0^1 h^\eta \left| k'\left(\left[hl^\mu + (1-h)n^\mu \right]^{\frac{1}{\mu}} \right) \right|^q dh \right)^{\frac{1}{q}} \right] \tag{49}$$

From the p-convexity of $\left|k'\right|^q$ and $\left|k'(l)\right| \le M$, equation (49) follows that

$$\begin{aligned}
\left|D_k(\eta,\mu;m,l,n)\right| &\le \frac{\left(l^\mu-m^\mu\right)^\eta}{n-m}\left[P^{\frac{1}{r}}(m)\left(\int_0^1 h^{\eta q+1}\left|k'\left(l^\mu\right)\right|^q dh+\int_0^1 h^{\eta q}(1-h)\left|k'\left(m^\mu\right)\right|^q dh\right)^{\frac{1}{q}}\right] \\
&+\frac{\left(n^\mu-l^\mu\right)^{\eta+1}}{n-m}\left[P^{\frac{1}{r}}(n)\left(\int_0^1 h^{\eta q+1}\left|k'\left(l^\mu\right)\right|^q dh+\int_0^1 h^{\eta q}(1-h)\left|k'\left(n^\mu\right)\right|^q dh\right)^{\frac{1}{q}}\right] \\
&\le \frac{\left(l^\mu-m^\mu\right)^\eta}{n-m}\left[P^{\frac{1}{r}}(m)\left(\left|k'\left(l^\mu\right)\right|^q\int_0^1 h^{\eta q+1}dh+\left|k'\left(m^\mu\right)\right|^q\int_0^1 h^{\eta q}(1-h)dh\right)^{\frac{1}{q}}\right] \\
&+\frac{\left(n^\mu-l^\mu\right)^{\eta+1}}{n-m}\left[P^{\frac{1}{r}}(n)\left(\left|k'\left(l^\mu\right)\right|^q\int_0^1 h^{\eta q+1}dh+\left|k'\left(n^\mu\right)\right|^q\int_0^1 h^{\eta q}(1-h)dh\right)^{\frac{1}{q}}\right] \\
&\le \frac{\left(l^\mu-m^\mu\right)^\eta}{n-m}\left[P^{\frac{1}{r}}(m)\left(M^q\frac{1}{\eta q+2}+M^q\frac{1}{(\eta q+1)(\eta q+2)}\right)^{\frac{1}{q}}\right] \\
&+\frac{\left(n^\mu-l^\mu\right)^{\eta+1}}{n-m}\left[P^{\frac{1}{r}}(n)\left(M^q\frac{1}{\eta q+2}+M^q\frac{1}{(\eta q+1)(\eta q+2)}\right)^{\frac{1}{q}}\right],
\end{aligned} \tag{50}$$

after simplification, equation (50) becomes

$$\left|D_k(\eta,\mu;m,l,n)\right| \le \frac{M}{n-m}\left(\frac{1}{\eta q+1}\right)^{\frac{1}{q}}\left[\left(l^\mu-m^\mu\right)^{\eta+1}P^{\frac{1}{r}}(m)+\left(n^\mu-l^\mu\right)^{\eta+1}P^{\frac{1}{r}}(n)\right].$$

Theorem 4

Let $k: I \subset (0,\infty) \to R$ be a differentiable mapping on I^o where $m,n \in I$ with $m<n$ such that $k' \in L[m,n]$. If $|k'|$ is p-convex on I and $\left|k'(l)\right| \le M \;\; \forall l \in \left[m, 2^{\frac{1}{\mu}}m\right]\left(if\; 2^{\frac{1}{\mu}}m<n, otherwise, l\in[m,n]\right)$, then the following identity holds

$$\begin{aligned}
\left|D_k(\eta,\mu;m,l,n)\right| &\le \frac{M}{n-m}\left(l^\mu-m^\mu\right)^{\eta+1}L^{1-\frac{1}{q}}(m)\left[R(m)+S(m)\right]^{\frac{1}{q}} \\
&+\frac{M}{n-m}\left(n^\mu-l^\mu\right)^{\eta+1}L^{1-\frac{1}{q}}(n)\left[R(n)+S(n)\right]^{\frac{1}{q}}
\end{aligned}$$

Where,

$$R(\xi)=\frac{\xi^{1-\mu}}{\eta+2}\,{}_2F_1\left(\eta+2,\frac{\mu-1}{\mu};\eta+3;1-\frac{l^\mu}{\xi^\mu}\right)$$

$$S(\xi)=\frac{\xi^{1-\eta}}{(\eta+1)(\eta+2)}\Bigg[(\eta+1)\,{}_2F_1\left(\eta+2,\frac{\mu-1}{\mu};\eta+3;1-\frac{l^{\mu}}{\xi^{\mu}}\right)$$

$$-(\eta+2)\,{}_2F_1\left(\eta+1,\frac{\mu-1}{\mu};\eta+2;1-\frac{l^{\mu}}{\xi^{\mu}}\right)\Bigg]$$

$$L(\xi)=\frac{\xi^{1-\mu}}{\eta+1}\,{}_2F_1\left(\eta+1,\frac{\mu-1}{\mu};\eta+2;1-\frac{l^{\mu}}{\xi^{\mu}}\right) \tag{51}$$

Proof

By using lemma and properties of modulus, it can be written as,

$$\left|D_k(\eta,\mu;m,l,n)\right|\le\frac{(l^{\mu}-m^{\mu})^{\eta}}{n-m}\int_0^1\frac{h^{\eta}\left|k'\left(\left[hl^{\mu}+(1-h)m^{\mu}\right]^{\frac{1}{\mu}}\right)\right|}{\left[hl^{\mu}+(1-h)m^{\mu}\right]^{1-\frac{1}{\mu}}}dh$$
$$+\frac{(n^{\mu}-l^{\mu})^{\eta+1}}{n-m}\int_0^1\frac{h^{\eta}\left|k'\left(\left[hl^{\mu}+(1-h)n^{\mu}\right]^{\frac{1}{\mu}}\right)\right|}{\left[hl^{\mu}+(1-h)n^{\mu}\right]^{1-\frac{1}{\mu}}}dh \tag{52}$$

by using power mean inequality, equation (52) can be written as,

$$\left|D_k(\eta,\mu;m,l,n)\right|\le\frac{(l^{\mu}-m^{\mu})^{\eta}}{n-m}\left[\left(\int_0^1 h^{\eta}\left(hl^{\mu}+(1-h)m^{\mu}\right)^{\frac{1}{\mu}-1}dh\right)^{1-\frac{1}{q}}\right.$$
$$\left.\left(\int_0^1 h^{\eta}\left(hl^{\mu}+(1-h)m^{\mu}\right)^{\frac{1}{\mu}-1}\left|k'\left(\left[hl^{\mu}+(1-h)m^{\mu}\right]^{\frac{1}{\mu}}\right)\right|^{q}dh\right)^{\frac{1}{q}}\right]$$
$$+\frac{(n^{\mu}-l^{\mu})^{\eta+1}}{n-m}\left[\left(\int_0^1 h^{\eta}\left(hl^{\mu}+(1-h)n^{\mu}\right)^{\frac{1}{\mu}-1}dh\right)^{1-\frac{1}{q}}\right.$$
$$\left.\left(\int_0^1 h^{\eta}\left(hl^{\mu}+(1-h)n^{\mu}\right)^{\frac{1}{\mu}-1}\left|k'\left(\left[hl^{\mu}+(1-h)n^{\mu}\right]^{\frac{1}{\mu}}\right)\right|^{q}dh\right)^{\frac{1}{q}}\right] \tag{53}$$

Let the integrals of equation (53) be,

$$I_1=\int_0^1 h^{\eta}\left(hl^{\mu}+(1-h)m^{\mu}\right)^{\frac{1}{\mu}-1}dh. \tag{54}$$

$$I_1=\int_0^1 h^{\eta}\left(hl^{\mu}+(1-h)n^{\mu}\right)^{\frac{1}{\mu}-1}dh. \tag{55}$$

Now solving the integrals I_1, from equation (55)

$$I_1=\int_0^1 \left(r^\eta\right)\left[rl^\mu+(1-r)m^\mu\right]^{\frac{1}{\mu}-1}dr$$

$$=\int_0^1 \left(r^\eta\right)\left[rl^\mu+m^\mu-rm^\mu\right]^{\frac{1}{\mu}-1}dr$$

$$=\left(m^\mu\right)^{\frac{1}{\mu}-1}\left(\int_0^1 r^{\eta+1-1}(1-r)^{\eta+2-(\eta+1)-1}\left[1-r\left(1-\frac{l^\mu}{m^\mu}\right)\right]^{-\left(1-\frac{1}{\mu}\right)}dr\right), \quad (56)$$

Multiplying and dividing equation (56) by $\beta(\eta+1,1)$,

$$I_1=\left(m^\mu\right)^{\frac{1}{\mu}-1}\frac{\beta(\eta+1,1)}{\beta(\eta+1,1)}\left(\int_0^1 r^{\eta+1-1}(1-r)^{\eta+2-(\eta+1)-1}\left[1-r\left(1-\frac{l^\mu}{m^\mu}\right)\right]^{-\left(1-\frac{1}{\mu}\right)}dr\right)$$

$$=\left(m^\mu\right)^{\frac{1}{\mu}-1}\beta(\eta+1,1)\left(\frac{1}{\beta(\eta+1,1)}\int_0^1 r^{\eta+1-1}(1-r)^{\eta+2-(\eta+1)-1}\left[1-r\left(1-\frac{l^\mu}{m^\mu}\right)\right]^{-\left(1-\frac{1}{\mu}\right)}dr\right), \quad (57)$$

Then by definition of Hypper-Goemetric function, equation (57) becomes,

$$I_1=\left(m^\mu\right)^{\frac{1}{\mu}-1}\beta(\eta+1,1)\left({}_2F_1\left(\eta+1,\frac{\mu-1}{\mu};\eta+2;1-\frac{l^\mu}{m^\mu}\right)\right)$$

$$=\left(m^\mu\right)^{\frac{1}{\mu}-1}\frac{\Gamma(\eta+1)\Gamma(1)}{\Gamma(\eta+1+1)}\left({}_2F_1\left(\eta+1,\frac{\mu-1}{\mu};\eta+2;1-\frac{l^\mu}{m^\mu}\right)\right)$$

$$=\left(m^\mu\right)^{\frac{1}{\mu}-1}\frac{1}{(\eta+1)}\left((\eta+2)\,{}_2F_1\left(\eta+1,\frac{\mu-1}{\mu};\eta+2;1-\frac{l^\mu}{m^\mu}\right)\right), \quad (58)$$

Then by equation (51), equation (58) becomes,

$$I_1=L(m). \quad (59)$$

By changing the variable 'm' into 'n' in equation (58), equation (55) becomes

$$I_2=\left(n^\mu\right)^{\frac{1}{\mu}-1}\frac{1}{(\eta+1)}\left((\eta+2)\,{}_2F_1\left(\eta+1,\frac{\mu-1}{\mu};\eta+2;1-\frac{l^\mu}{n^\mu}\right)\right), \quad (60)$$

Then by equation (51), equation (60) becomes,

$$I_2=L(n). \quad (61)$$

By equation (59) and (61), equation (53) becomes

$$\left|D_k(\eta,\mu;m,l,n)\right| \le \frac{\left(l^\mu-m^\mu\right)^\eta L^{1-\frac{1}{q}}}{n-m}\left(\int_0^1 h^\eta\left(hl^\mu+(1-h)m^\mu\right)^{\frac{1}{\mu}-1}\left|k'\left(\left[hl^\mu+(1-h)m^\mu\right]^{\frac{1}{\mu}}\right)\right|^q dh\right)^{\frac{1}{q}}$$
$$+\frac{\left(n^\mu-l^\mu\right)^{\eta+1} L^{1-\frac{1}{q}}}{n-m}\left(\int_0^1 h^\eta\left(hl^\mu+(1-h)n^\mu\right)^{\frac{1}{\mu}-1}\left|k'\left(\left[hl^\mu+(1-h)n^\mu\right]^{\frac{1}{\mu}}\right)\right|^q dh\right)^{\frac{1}{q}} \quad (62)$$

From the p-convexity of $\left|k'\right|^q$, equation (62) follows that

$$\left|D_k(\eta,\mu;m,l,n)\right| \le \frac{\left(l^\mu-m^\mu\right)^\eta L^{1-\frac{1}{q}}}{n-m}\left(\int_0^1 h^\eta\left(hl^\mu+(1-h)m^\mu\right)^{\frac{1}{\mu}-1}\left|k'(l)\right|^q dh\right.$$
$$\left.+\int_0^1\left(h^\eta-h^{\eta+1}\right)\left(hl^\mu+(1-h)m^\mu\right)^{\frac{1}{\mu}-1}\left|k'(m)\right|^q dh\right)^{\frac{1}{q}}$$
$$+\frac{\left(n^\mu-l^\mu\right)^{\eta+1} L^{1-\frac{1}{q}}}{n-m}\left(\int_0^1 h^\eta\left(hl^\mu+(1-h)n^\mu\right)^{\frac{1}{\mu}-1}\left|k'(l)\right|^q dh\right. \quad (63)$$
$$\left.+\int_0^1\left(h^\eta-h^{\eta+1}\right)\left(hl^\mu+(1-h)n^\mu\right)^{\frac{1}{\mu}-1}\left|k'(n)\right|^q dh\right)^{\frac{1}{q}}$$

Then by equation (30), (34), (37) and (38), equation (63) becomes

$$\left|D_k(\eta,\mu;m,l,n)\right| \le \frac{\left(l^\mu-m^\mu\right)^\eta L^{1-\frac{1}{q}}}{n-m}\left(\left|k'(l)\right|^q R(m)+\left|k'(m)\right|^q S(m)\right)^{\frac{1}{q}}$$
$$+\frac{\left(n^\mu-l^\mu\right)^{\eta+1} L^{1-\frac{1}{q}}}{n-m}\left(\left|k'(l)\right|^q R(n)+\left|k'(n)\right|^q S(n)\right)^{\frac{1}{q}}, \quad (64)$$

then by boundedness of $\left|k'(l)\right|$, equation (64) becomes

$$\left|D_k(\eta,\mu;m,l,n)\right| \le \frac{\left(l^\mu-m^\mu\right)^\eta L^{1-\frac{1}{q}}}{n-m}\left(M^q\left(R(m)+S(m)\right)\right)^{\frac{1}{q}}$$
$$+\frac{\left(n^\mu-l^\mu\right)^{\eta+1} L^{1-\frac{1}{q}}}{n-m}\left(M^q\left(R(n)+S(n)\right)\right)^{\frac{1}{q}}, \quad (65)$$

after simplification, equation (65) becomes

$$\left|D_k(\eta,\mu;m,l,n)\right| \le M\frac{\left(l^\mu-m^\mu\right)^\eta L^{1-\frac{1}{q}}}{n-m}\left(R(n)+S(n)\right)^{\frac{1}{q}}$$
$$+M\frac{\left(n^\mu-l^\mu\right)^{\eta+1} L^{1-\frac{1}{q}}}{n-m}\left(R(n)+S(n)\right)^{\frac{1}{q}}.$$

So the above equation completes the theorem.

Conclusion

In the above working, a few lemmas and results are sighted to get the now upper bounds for the inequalities of Ostrowski type via the means of fractional order integral operators of Katugampola type. The new lemmas and results can also be get from the same methodology which used in above mentioned theorems and Lemmas.

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