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Minimal equitable dominating graph

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Abstract
Many generalizations, results and new classes of graphs are obtained in this paper. We defined a new class of intersection graphs which are obtained by usage equitable dominating sets. We introduce the concept of minimal equitable dominating graphs and characterization of graphs are obtained.

Keywords: Equitable domination, minimal equitable domination graph

Introduction

Sampathkumar introduced the concept of equitable domination. In ordinary domination mere adjacency between vertices is enough for a vertex to dominate another. In practice, neighbourliness alone between persons is not enough for friendship. If the persons have nearly equal status then it may result for friendship. Human beings have a tendency to move with others only when they have either nearly equal wealth or equal educational qualification or equal status in society. A graph model for this practical situation has to take care of this equitability. That is vertices have to be assigned labels which can be compared, fro example, positive integers. If labels of two vertices are almost equal then they we may say that, vertices are equitable. This leads to an elegant domination concept namely minimal equitable dominating graph.

A subset D of V (G) is called an equitable dominating set of a graph G if for every $v \in (V - D)$, there exists a vertex $v \in D$ such that $uv \in E(G)$ and $|d(u) - d(v)| \leq 1$. The minimum cardinality of such a dominating set is denoted by γ_e and is called equitable domination number of G. A equitable dominating set D is said to be minimal if for any vertex $u \in D$, $D - \{u\}$ is not a equitable dominating set of G. A subset S of V is called a equitable independent set, if for any $u \in S$, $v \notin N_e(u)$, for all $u \in S - \{u\}$ the equitable neighbourhood of u denoted by $N_e(u)$ is defined as $N_e(u) = \{ v \in N(u), |d(u) - d(v)| \leq 1\}$ and $u \in I_e$ if and only if $N_e(u) = \phi$. The cardinality of $N_e(u)$ is denoted by $d_e(u)$. the maximum and minimum equitable degree of a point in G are denoted respectively by $\Delta_e(G)$ and $\delta_e(G)$. That is $\Delta_e(G) = \max_{u \in V(G)} |N_e(u)|$, $\delta_e(G) = \min_{u \in V(G)} |N_e(u)|$. If a vertex $u \in V$ be such that $|d(u) - d(v)| \geq 2$ for all $v \in N(u)$ then u is in equitable dominating set. Such points are called equitable isolates.

Many intersection graphs are introduced by Kulli and Janakiram. They studied a new class of intersection graphs in the field of domination. As the minimal dominating sets are the vertices of minimal dominating graph and any two vertices are adjacent if the intersection of any two set are not empty. This motivated us to introduce the concept of minimal equitable dominating graphs. Minimal equitable dominating graph (MED(G)) of a graph G is the intersection graph defined on the family of all minimal equitable dominating sets of vertices in G. Any two vertices in MED(G) are adjacent if they have vertex in common in minimal equitable dominating sets of G. In this paper we study some interesting properties of these graphs. We get the characterization for the graphs where its MED(G) are connected and also those graphs in which MED(G) is complete graph. We illustrate the minimal equitable dominating graph by the following example.

Example 1.1.

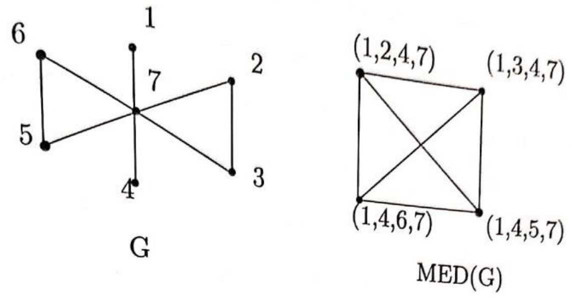


Fig 1

2. Main Results

Observation 2.1. In any graph G, any maximal equitable independent set is also minimal equitable dominating set.

Observation 2.2. Let $G = K_{m,n.}$ If $|m - n| \geq 2$, then MED (G) $\cong K_1$.

Theorem 2.3. If G is a graph without equitable isolated vertices, then for any minimal equitable dominating set D, V – D is equitable dominating set.

Proof. Let G = (V, E) be a graph without equitable isolated vertex. D be minimal equitable dominating set, we prove that V – D is also minimal equitable dominating set. From Theorem in [5] that if D is dominating set then V – D is a dominating set. Let x be any vertex in D then there is some vertex y in V – D which is adjacent and equitable degree with x. That is V – D is equitable dominating set.

Converse is proved by contradiction method. Supposing that V – D is not minimal dominating set, we get a contradiction.

Theorem 2.4. For any graph G without equitable isolated vertices and $p \geq 2$ vetices, MED (G) is connected if and only if $\Delta_e(G) < p - 1$.

Proof. Let $\Delta_e(G) < p - 1$. Let D_1, D_2 be two disjoint minimal equitable dominating set of G. We consider the following cases:

Case (i). Suppose there exist two vertices $u \in D_1$ and $v \in D_1$ such that u and v are not adjacent. Then there exists a maximal independent set D_3 containing u and v. Since D_3 is also a minimal dominating set, this implies that D_1 and D_2 are connected in MED (G) through D_3.

Case (ii). Suppose every vertex in D_1 is adjacent to every vertex in D_2. We consider the following two subcases.

a. Suppose there exists two vertices $u \in D_1$ and $v \in D_1$ such that every vertex not in $D_1 \cup D_2$ is adjacent to either u or v. Then {u, v} is a minimal equitable dominating set of G. Hence D_1 and D_2 are connected in MED (G) through {u,v}.
b. Suppose for any two vertices $u \in D_1$ and $v \in D_1$, there exists a vertex $w \notin D_1 \cup D_2$ such that w is adjacent to neither u nor v. Then there exist two minimal equitable independent sets D_3 and D_4 containing u, w and v respectively. Thus, as above, D_1 and D_2 are connected in MED (G) through D_3 and D_4.

Thus, from the above cases, every two vertices in MED (G) are connected and hence MED (G) is connected.

Conversely, suppose MED (G) is connected. Assume $\Delta_e(G) < p - 1$ and u is a vertex degree $(p - 1)$. Then $\{u\}$ is a minimal equitable dominating set of G and $V - \{u\}$ also contains a minimum equitable dominating set of G. This shows that MED (G) has at least two components, a contradiction. Hence $\Delta_e(G) < p - 1$.

Theorem 2.5. For any graph G, MED (G) is either connected or has at least one component isomorphic to K_1.

Proof. By assuming the maximal equitable of a graph G. $\Delta_e(G)$, we have two probabilities:

Case (i). $\Delta_e(G) < p - 1$ and from Theorem 2.4 G will be connected.

Case (ii). $\Delta_e(G) < p - 1$, in this case we have two subcases:

a. $\delta_e(G) = p - 1$, that is G is complete graph with p vertices and every vertex in the complete graph is MED set. Then there exists one component in MED (G) isomorphic to K_1.
b. $\delta_e(G) < \Delta_e(G) = p - 1$, suppose $\{u_1, u_2, \ldots u_t\}$ is the vertices in G which $d_e(u_i) = p - 1$, where i=1, 2, …, t. Let H be a subgraph, $H = \{u_1, u_2, \ldots u_t\}$. It is clear that $\Delta_e(G) < |H| -1$. By Theorem 2.4, MED (H) is connected, and every $\{u_i\}$ is MED set of G and not connected to other MED (G). Then MED (G) contains at least one component isomorphic to K_1.

Theorem 2.6. The graph G is complete if and only if

Proof: MED (G) = $\overline{K_p}$, since $G = K_p$ then for any vertex u, the set $\{u\}$ is minimal equitable dominating set for all $\{u\}$ then there are p disjoint MED sets.

Hence MED (G) = $\overline{K_p}$.

Conversely, MED (G) = $\overline{K_p}$then we prove that G is complete.

Let us suppose that $G \neq K_p$. Then there exists at least two vertices (u, v) which are not adjacent in G. Then there exists a maximum equitable independent set say D′ containing u and v, that by Observation 1 D will be MED set and its size at least 2. Since every MED set of $\overline{K_p}$ is of size one, a contradiction.

Or MED (G) = $\overline{K_p} \Rightarrow \gamma_e$ MED (G) = p by Theorem 2.4, we have $G = K_p$.

Theorem 2.7. For any graph G, MED (G) is complete if and only if G contains an equitable isolated vertex.

Proof. Let G be a graph and MED (G) is complete. That is for any two minimum equitable dominating set the intersection is not empty. So that there exist at least one common vertex between all the minimal equitable dominating set should be equitable isolated vetex. Hence G has at least one isolated vertex.

If there is equitable isolated vertex in G, then this vertex should be in all the minimal equitable dominating set. That is MED (G) is complete.

Theorem 2.8. If G is a complete graph K_p, then γ_e (MED (G)) = p.

Proof. If G is complete graph then the MED (G) is totally disconnected and there is p minimal equitable dominating sets. Then γ_e (MED (G)) = γ (MED (G)) = p.

Theorem 2.9. Let $G = 2\ K_r$. Then MED (G) $\cong K_r \times K_r$.

Proof. Let G be a graph $G = 2K_r$, then all the MED set contains two points one from each copy. That is there is r^2 MED set. This graph is isomorphic to $K_r \times K_r$ from the definition of Cartesian product. We can generalize the Theorem 2.9 by the following.

Theorem 2.10. Let G be a disjoint union of two complete graphs K_m and K_n. Then the MED (G) is isomorphic to the cartsian product of K_m and K_n.

Proof. From the definition of Cartesian product the vertices of MED (G) are the same vertices of the graph K_m x K_n. Let $\{x, y\}$ and $\{x', y'\}$ be two minimal equitable dominating sets in G. Let F be a function defined by

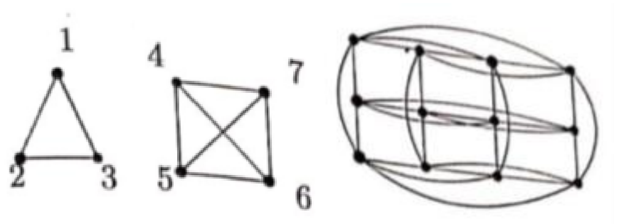
F: V (MED (G)) $\rightarrow$ V (K_m x K_n),

such that $f(\{x,y\}) = (x,y)$. To prove that MED (G) $\cong K_m$ x K_n, we need to show that the function F: V (MED (G)) $\rightarrow$ V (K_m x K_n), is isomorphic.

The function F is one-one and onto because if (x,y) be any vertices in V (K_m x K_n) then there is only one vertex in V (MED(G)) such that $f(\{x,y\}) = (x,y)$. Also is onto because for any (x,y) in V (K_m x K_n) there exists minimal equitable dominating set $\{x, y\}$ such that $f(\{x,y\}) = (x,y)$.

Now we prove that the function F: V (MED (G)) $\rightarrow$ V (K_m x K_n) preserve the adjacency and non-adjacency. Let D = (x,y) and D′ = (x', y') be two minimal equitable dominating sets which are adjacent in MED (G) that is D $\cap$Dr, $\neq \theta$. Hence either $x = x'$ or $y = y'$. Suppose without loss of generality $x - x'$ then y is adjacent to y' since x and x' are in K_m and y and y' are in K_n. Thus from the definition of Cartesian product, we have (x,y) is adjacent to $(x' y')$.

Example 2.11. Let G be a graph as in figure 2a be the disjoint union of K_3 and K_4. That is G = $K_3 \cup K_4$. The minimal equitable dominating sets of G are {1, 4}, {1, 5}, {1, 6}, {1, 7}, {2, 4}, {2, 5}, {2, 6}, {2, 7}, {3, 4}, {3, 5}, {3, 6} and {3. 7}. And hence the MED (G) as in figure 2b which is isomorphic to K_3 x K_4.



2A 2B

Fig 2

Theorem 2.12. γ_e(MED(G)) = 1 if G is totally disconnected or G $\cong K_{1,r}$, for r > 3.

Proof. If G is totally disconnected graph then there exists only one MED set which contains all the vertices of the graph G.

Then γ_e(MED(G)) = 1

Similarly if G $\cong K_{1,p}$ we get MED (G) contain one point.

Lemma 2.13. For any graph G, $d_e(G) \leq \delta_e(G) + 1$.

Proof. Let d_e (G) = |F| such that F is the set of maximum partition of dominating set of G. Suppose if it is possible $d_e(G) > \delta_2(G) + 1$.

Then there exists at least one vertex $u \in$ V (G) such that $d_e(u) = \delta_e(G)$. That implies

$|N_e(u)| = \delta_e(G)$.

$\Rightarrow |N_e[u]| = \delta_e(G) + 1$

$\Rightarrow d_e(G) = |N_e[u]|$.

Hence at least every dominating set from F has at least one element from the set $N_e(u)$, but the number of the sets in F is greater than $\delta_e(G) + 1$. That is there exists at least one dominating set in F not containing any element from $N_e(u)$, implies u is not dominated by some dominating set, a contradiction.

Theorem 2.14. For any graph G, β_e(MED(G)) = d_e(G).

Proof. Let F be a maximum order equitable domatic partition of V(G). Where F – {$D_1, D_2, \ldots, D_d$} such that d_e (G) = d, we know that for any graph G every equitable dominating set D, either D is MED set or contains a MED set. That is for any dominating set D_i in F there exists $D_i' \subseteq D_i$ for I = 1,2,…,d.

We now define, F′ = {D_i', D_2',…,D_d'}, where D_i' is MED set of G and $D_i' \subseteq D_i$.

|F′| = d = d_e (G)

And F′ is maximum equitable independent dominating set for MED (G).

Hence β_e(MED(G)) = d_e(G).

Corollary 2.15. For any graph G, $\beta_e(G) \leq$ | V (MED(G)) |

Proof. for any graph G_1,

$\beta_e(G) \leq p = |V(G)|$,

$\beta_e(MED(G)) \leq |V(V(MED(G))|$,

$d_e(G) = \beta_e(MED(G)) \leq |V(MED(G))|$,

Hence, $d_e(G) \leq |V(MED(G))|$,

Corollary 2.16. For any graph G, γ_e(MED)) $\leq \delta_e(G) + 1$. Further the equality holds if G is complete.

Proof. Let G = (V, E) be a graph. We have for any graph G,

γ_e(MED)) $\leq \beta_e$(MED(G))

$\Rightarrow$ By Theorem (2.13), $\beta_e(MED(G)) \leq \delta_e(G)$

$\Rightarrow$ By Lemma (2.12), $\gamma_e(MED(G)) \leq \delta_e(G) + 1$.

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