International Journal of Multidisciplinary Research and Growth Evaluation

International Journal of Multidisciplinary Research and Growth Evaluation
ISSN: 2582-7138
Received: 08-04-2021; Accepted: 26-04-2021
www.allmultidisciplinaryjournal.com
Volume 2; Issue 3; May-June 2021; Page No. 108-109

A short proof of anti-Ramsey number for cycles

Noorya Yousifi
Lecturer, Department of Mathematics, Education Faculty, Jawzjan University, Afghanistan

Corresponding Author: **Noorya Yousifi**
DOI: https://doi.org/10.54660/.IJMRGE.2021.2.3.108-109

Abstract
Ramsey's theorem states that there exists a least positive integer R(r, s) for which every blue-red edge colouring of the complete graph on R(r, s) vertices contains a blue clique on r vertices or a red clique on s vertices. This work contains a simplified proof of Anti-Ramsey theorem for cycles. If there is an edge e between H and H0, incident to, say, some $v \in H$ of color from NEWc(v), then we can make H and H0 connected by adding the edge e and deleting some edge incident to v of the same color as e in H, so the resulting graph G˜ has a connected component of order $\geq 2($ k+1/2), which contradicts that every connected component is of order $\leq k - 1$. Since each component is Hamiltonian and of order $\geq$ k+1/ 2 , to avoid a rainbow Ck, by the same type of argument as in Claim 1, we must have that |c(H, H0)| = 1.

Keywords: Ramsey’s theorem, anti-ramsey, edge colouring

Introduction

For a graph H, the classical anti-Ramsey number AR(n, H) is the maximum number of colors in a coloring of edges of Kn with no rainbow copy of H. It was introduced by Erdos, Simonovits and S´os [1975]. When H is a cycle of length k, Ck, they provided a rainbow Ck-free coloring of edges of Kn with n {k−2/2 + 1/k−1} + O(1) colors and conjectured that this is optimal. They proved the conjecture for k = 3. Alon [1] proved the conjecture for k = 4 and derived an upper bound for general k. In [4], Jiang and West improved the general upper bound and mentioned that the conjecture has been proven for $k \leq 7$. Finally, Montellano-Ballesteros and Neumann-Lara (2005) proved the conjecture completely. The main technique used in the previous study is a careful, detailed analysis of a graph representing the coloring, in particular, proving that each component of such a graph is Hamiltonian if each vertex has enough “new” colors. This paper uses the same idea as in earlier study, but shortens the proof.

Theorem 1. For $k \geq 3$ and $n \in N$,
AR(n, Ck) $\leq$ n {k – 2/2 + 1/k – 1} − 1

Definitions and proofs of main results

Let K = Kn for a fixed n. For an edge coloring c of K, and a vertex $v \in V$ (K), let the set of new colors at v, NEWc(v), be the set of colors used on edges between v and V (K)\{v}, but not used on edges spanned by V (K) \ {v}. Let newc(v) = |NEWc(v)|. Then the number of colors used by c on K, |c|, equals newc(v) + |c(K − v)|, where for a subgraph H of K, |c(H)| denotes the number of colors used by c on the edges of H. Here we simply have written |c| instead of |c(K)|. For pairwise disjoint subsets X, Y, Z of V (K), let K[X] be the subgraph induced by X, K[X, Y] the bipartite subgraph induced by X and Y , K[X, Y, Z] the tripartite subgraph induced by X, Y , and Z. Then the corresponding sets of colors used in those subgraphs are denoted by c(X), c(X, Y), and c(X, Y, Z) respectively. For a subgraph H of a graph G and a vertex v of G, let degH(v) := |NG(v) $\cap$ V (H)|. We now state a version of the Dirac and Ore’s theorems for Hamiltonian cycle which is essential for our proofs.

Theorem 2 (Dirac, 1952; Ore, 1960). Let P = v1, v2, . . . , vm, $m \geq 3$, be a path in a connected graph G. Suppose degP (v1) + degP (vm) $\geq$ m.

1. Then V (P) contains a cycle of length m in G.
2. If P is a longest path in G, then G is Hamiltonian.

We define a few special edge colorings of a complete graph with no rainbow Ck. We say that an edge-coloring c of K is weak k-anticyclic if there is a partition of V (K) into sets V1, . . . , Vt with 1 ≤ |Vi | ≤ k − 1, i = 1, . . . , t, such that (i) for any i, j with 1 ≤ i < j ≤ t, |c(Vi , Vj)| = 1; (ii) for any i, j, ` with 1 ≤ i < j < ` ≤ t, |c(Vi , Vj , V`)| ≤ 2; and (iii) c has no rainbow Ck. In addition, if all but at most one of the parts of the partition are exactly of size k − 1 and the edges spanned by each Vi have own colors (i.e., colors used only once), then c is called k-anticyclic.

We denote a fixed coloring from the set of k-anticyclic colorings of Kn such that the color of any edge between Vi and Vj is min{i, j}, by c ∗ . Then we easily see the following.

Lemma 2.2. If c is weak k-anticyclic, then

|c| ≤ |c ∗ | ≤ n { k – 2/2 + 1/k – 1} − 1.

Next lemma is the main tool for the proof of the main theorem. It appears in a different form in [Montellano-Ballesteros and Neumann-Lara (2005), Lemma 9]. We include it here for completeness.

Lemma 2.2. Let k ≥ 4. Let c be an edge-coloring of K with no rainbow Ck. If for all x, y ∈ V (K) with x 6= y, newc(x) ≥ 2 and newc(x) + newc(y) ≥ k − 1, (3.1) then c is weak k-anticyclic. Proof. Consider a representing graph G of c such that it spans K and has exactly one edge of each color from {NEWc(v)| v ∈ V (K)}. The hypothesis (3.1) gives a bound on degrees of vertices in G, namely the sum of degrees of two distinct vertices in G is at least k − 1. In the following, H denotes a connected component of G.

Claim 1 If there is a cycle of length k − 1 in H, then |V (H)| = k − 1. ------------ (1.1)

Suppose not, i.e., there is a cycle, (v1, . . . , vk−1, v1), and V (H) \ {v1, . . . , vk−1} 6= ∅. Since H is connected, some u ∈ V (H) \ {v1, . . . , vk−1} is adjacent to some vertex in {v1, . . . , vk−1}, say v1. If c(u, v1) ∈ NEWc(v1), then c(u, v2) = c(v2, v3); otherwise (v1, u, v2, v3, . . . , vk−1, v1) in K is a rainbow Ck. Similarly c(u, v3) = c(v3, v4), . . . , c(u, vk−1) = c(vk−1, v1), and eventually c(u, v1) = c(v1, v2), which contradicts that uv1 and v1v2 are edges of H. Hence c(u, v1) ∈ NEWc(u). By the similar argument as above, we have c(u, {v1, . . . , vk−1}) = c(u, v1) ∈ NEWc(u). Since we assumed newc(u) ≥ 2, there is w ∈ V (H)\ {v1, . . . , vk−1, u} with c(u, w) ∈ NEWc(u) and c(u, w) 6= c(u, v1). Considering cycles of length k in K, (vk−2, u, w, v1, v2, . . ., vk−2) and (v1, w, u, v3, . . . , vk−1, v1), we have c(w, v1) = c(v1, v2) = c(vk−1, v1), which contradicts that v1v2 and vk−1v1 are edges of H.

Claim 2 k+1/2 ≤ |V (H)| ≤ k − 1 ((Hence H is Hamiltonian from by (1.1) and Theorem 1

The lower bound follows from (3.1). If in H every path has at most k − 1 vertices or there is a cycle of length k − 1, then from Theorem 1 and Claim 1, we have that the upper bound holds. Hence we may assume that in H there is a path on at least k vertices, but no Ck−1. In particular, we can find a path, v1, . . . , vk satisfying c(vk−1, vk) ∈ NEWc(vk−1) since (i) considering P1 := v1, . . . , vk−1, to avoid Ck−1, we have degP1 (v1) + degP1 (vk−1) < k − 1; (ii) from (1.1), without loss of generality we can find a vk ∈ V (H) \ V (P1) such that vk−1vk is an edge of H and c(vk−1, vk) ∈ NEWc(vk−1). Let P2 := v2, . . . , vk. Then degP2 (v2) ≥ newc(v2), (3.2) since otherwise there is x ∈ V (H) \ V (P2) such that c(x, v2) ∈ NEWc(v2) and c(x, v2) 6= c(v2, v3), in which case we obtain a rainbow Ck in K, namely (x, v2, . . . , vk, x). Also we have degP2 (vk) < newc(vk) since otherwise together with (3.2), V (P2) induces a cycle of length k −1 in H by Theorem 1. Therefore we can find a vk+1 ∈ V (H) \ V (P2) such that vkvk+1 is an edge of H and c(vk, vk+1) ∈ NEWc(vk). Note that vk+1 6= v2 since otherwise (v2, . . . , vk, v2) is a rainbow Ck−1 in H. Let P3 := v3, . . . , vk, vk+1. Then degP3 (v3) ≥ newc(v3), (3.3) since otherwise there is y ∈ V (H) \ V (P3) such that c(y, v3) ∈ NEWc(v3) and c(y, v3) 6= c(v3, v4), so (y, v3, . . . , vk+1, y) is a rainbow Ck in K. Now we note that c(v2, vk+1) = c(v2, v3) to avoid a rainbow Ck induced by {v2, . . . , vk+1} in K. Let S = {i + 1 | v2vi ∈ E(H), i = 3, . . . , k − 1} and T = {j | v3vj ∈ E(H), j = 4, . . . , k}. So S, T ⊆ {4, . . . , k} and |S| + |T| ≥ newc(v2) + newc(v3) ≥ k − 1. Thus |S ∩ T| ≥ 2. Let i + 1 ∈ S ∩ T where i 6= 3. Then (v2, vi , vi−1, . . . , v3, vi+1, vi+2, . . . , vk+1, v2) is a rainbow Ck

Claim 3 For any two components H and H0 of G, |c(H, H0)| = 1

If there is an edge e between H and H0, incident to, say, some v ∈ H of color from NEWc(v), then we can make H and H0 connected by adding the edge e and deleting some edge incident to v of the same color as e in H, so the resulting graph G˜ has a connected component of order ≥ 2(k+1/2), which contradicts that every connected component is of order ≤ k − 1. Hence the colors of edges between H and H0 are not from c(H) nor from c(H0). Since each component is Hamiltonian and of order ≥ k+1/ 2 , to avoid a rainbow Ck, by the same type of argument as in Claim 1, we must have that |c(H, H0)| = 1

References

1. Alon N. On a conjecture of Erd˝os, Simonovits, and S´os concerning anti-Ramsey theorems, J. Graph Theory. 1983; 1:91-94.
2. Dirac GA. Some theorems on abstract graphs, Proc. Lond. Math. Soc. 1952; 2:69-81.
3. Erd˝os P, Simonovits M, S´os VT. Anti-Ramsey theorems, Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erd˝os on his 60th birthday), Vol. II, pp. 633–643. Colloq. Math. Soc. Janos Bolyai, Vol. 10, North-Holland, Amsterdam, 1975.
4. Jiang T, West D. On the Erd˝os-Simonovits-S´os Conjecture on the anti-Ramsey number of a cycle, Combin. Probab. Comput. 2003; 12:585-598.
5. Montellano-Ballesteros JJ, Neumann-Lara V. An anti-Ramsey theorem on cycles, Graphs Combin. 2005; 21(3):343-354.
6. Ore O. A Note on Hamiltonian Circuits, American Mathematical Monthly. 1960; 67:55.
7. Schiermeyer I. Rainbow 5- and 6-cycles: A proof of the conjecture of Erd˝os, Simonovits and S´os, Preprint, TU Bergakademie Freiberg, 2001.