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On the non-homogeneous sextic equation with three unknowns $3(x^2 + y^2) - 5xy = 36z^6$

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Abstract

The sextic non-homogeneous equation with four unknowns represented by the Diophantine equation $3(x^2 + y^2) - 5xy = 36z^6$ is analyzed for its patterns of non-zero distinct integral solutions are illustrated. Various interesting relations between the solutions and special numbers, namely polygonal numbers, Pyramidal numbers, Jacobsthal numbers, Jacobsthal-Lucas number are exhibited.

Keywords: Integral solutions, sextic, non-homogeneous equation

1. Introduction

In Number theory, Diophantine equations play a significant role and have a marvelous effect on credulous people .They occupy a remarkable position due to unquestioned historical importance. The subject of Diophantine equations is quite difficult. Every century has seen the solutions of more mathematical problems than the previous century and yet many mathematical problems ,both major and minor, still remains unsolved. It is hard to tell if a given equation has solutions or not and when it does, there may be no method to find all of them. It is difficult to tell which are easily solvable and which require advanced techniques. There is no well unified body of knowledge concerning general methods. A Diophantine problem is considered as solved if a method is available to decide whether the problem is solvable or not and in case of its solvability, to exhibit all integers satisfying the requirements set forth in the problem. The successful completion of exhibiting all integers satisfying the requirements set forth in the problem add to further progress of Number theory as they offer good applications in the field of Graph theory ,Modular theory ,Engineering, Music ,Coding and Cryptography and so on. Integers have repeatedly played a crucial role in the evolution of the natural sciences. The theory of integers provide answers to real world problems. The theory of Diophantine equations offers a rich variety of fascinating problems [1-4]. Particularly, in [5-10], sextic equations with three unknowns are studied for their integral solutions [11-17]. Analyze sextic equations with four unknowns for their non-zero integer solutions [18-23] analyze sextic equations with five unknowns for their non-zero solutions. This communication concerns with yet another interesting a non-homogeneous sextic equation with three unknowns given by $3(x^2 + y^2) - 5xy = 36z^6$. Infinitely many non-zero integer tuple (x,y,z) satisfying the above equation are obtained. Various interesting properties among the values of x,y,z are presented.

Notations

$t_{m,n}$ **:** Polygonal number of rank n with size m

CP_n^m **:** Centered Pyramidal number of rank n with size m

P_n^m : Pyramidal number of rank n with size m

j_n : Jacobsthal lucas number of rank n

J_n: Jacobsthal number of rank n

KY_n: Kynea number of rank n

$F_{4,n,m}$: Four dimensional figurate number of rank n with size m

2. Method of analysis

The non-homogeneous sextic equation to be solved for its non-zero integral solution is

$$3(x^2+y^2)-5xy=36z^6 \qquad (1)$$

Introduction of the linear transformations,

$$x=u+v,\ y=u-v \qquad (2)$$

In (1) leads to

$$u^2+11v^2=36z^6 \qquad (3)$$

Different methods of obtaining the patterns of integer solutions to (1) are illustrated below:

2.1 Pattern.1

$$\text{Let } z=a^2+11b^2 \qquad (4)$$

36 can be written as

$$36=\left(5+i\sqrt{11}\right)(5-i\sqrt{11}) \qquad (5)$$

Using (5), (4) in (3) and applying the method of factorization, define

$$u+i\sqrt{11}v=\left(5+i\sqrt{11}\right)(\alpha+i\sqrt{11}\beta)$$

Where

$$\alpha=a^6-165a^4b^2+1815a^2b^4-1331b^6$$
$$\beta=6a^5b-220a^3b^3+726ab^5$$

Equating real and imaginary parts, we get

$$\left.\begin{aligned}u&=5\alpha-11\beta\\ v&=\alpha+5\beta\end{aligned}\right\} \qquad (6)$$

Using (6) and (2) we have

$$\left.\begin{aligned}x(a,b)&=6(\alpha-\beta)\\ y(a,b)&=4(\alpha-4\beta)\end{aligned}\right\} \qquad (7)$$

Thus, (4) and (7) represent the integer solutions to the equation (1).

2.2 Properties

(i)$4x(n,1)-6y(n,1)=240CP_n^6\left(110-3t_{4,n}\right)+240(726t_{3,n}-363t_{4,n})$

(ii) $z(2^n+1,2^n)=KY_n+11j_{2n}-9$

(iii)$8x(1,n)-36y(1,n)=36\left(1-165t_{4,n}\right)+36t_{4,n}^2(1815-1331t_{4,n})$

2.3 Pattern: 2

Consider (3) as

$$u^2+11v^2=36z^6*1 \qquad (8)$$

Write 1 as

$$1=\frac{(5+i\sqrt{11})(5-i\sqrt{11})}{36} \qquad (9)$$

Substituting (4) and (9) in (8) and employing the factorization method, define

$$u+i\sqrt{11}v=\frac{\left(5+i\sqrt{11}\right)^2}{6}\left(\alpha+i\sqrt{11}\beta\right)$$

Equating real and imaginary parts, we get

$$\left.\begin{aligned}u&=\frac{1}{6}[14\alpha-110\beta]\\ v&=\frac{10\alpha+14\beta}{6}\end{aligned}\right\} \qquad (10)$$

As our interest is on finding integer solutions, we choose $\alpha\ and\ \beta$ suitably so that u and v are integers.

Using (10) in (2) and replacing a by 6a and b by 6b, the corresponding integer solutions to (1) are found to be

$$x(a,b)=6^5(24\alpha-96\beta)$$
$$y(a,b)=6^5(4\alpha-124\beta)$$
$$z(a,b)=\ 6^2(a^2+11b^2)$$

2.4 Pattern: 3

(3) can be written as

$$u^2+11v^2=(6z^3)^2$$

Which is satisfied by

$$\left.\begin{aligned}v&=72PQ\\ u&=36(11P^2-Q^2)\end{aligned}\right\} \qquad (11)$$

$$z^3=66P^2+6Q^2 \qquad (12)$$

To find z, assume

$$z=66a^2+6b^2 \qquad (13)$$

In (12), employing the method of factorization and performing a few calculations, we obtain

$$\left.\begin{aligned}P&=66a^3-18ab^2\\ Q&=\ 198a^2b-6b^3\end{aligned}\right\} \qquad (14)$$

Substituting (14) in (11) and using (2), we have

$$\left.\begin{aligned}x(a,b)&=396\,P^2-36Q^2+72PQ\\ y(a,b)&=396P^2-36Q^2-72PQ\end{aligned}\right\} \qquad (15)$$

Thus, (13) and (15) represent the non-zero distinct integral solutions to (1).

Note:1

It is seen that in addition to (5), 36 may also be written in two different ways as

$$\left.\begin{aligned}(i)36 &= \frac{(7+i5\sqrt{11})(7-i5\sqrt{11})}{9} \\ (ii)36 &= \frac{(3+i9\sqrt{11})(3-i9\sqrt{11})}{25}\end{aligned}\right\} \quad (16)$$

Following the procedure similar to pattern.1, the corresponding non-zero integer solutions and properties for the above two cases are as follows:

Solutions for (i)

$$x(a,b) = 3^5(12\alpha - 48\beta)$$
$$y(a,b) = 3^5(2\alpha - 62\beta)$$
$$z(a,b) = 3^2(a^2 + 11b^2)$$

2.5 Properties

(i) $x(n,1) - 6y(n,1)$
$$= 3^5 * 324\,[-CP_n^6(6t_{4,n} - 220) + 1452t_{3,n} - 726t_{4,n}]$$
(ii) $z(2^n - 1, 2^n) = 3^2[12j_{2n} - j_{n+1} - 11 + (-1)^{n+1}]$

$(iii)31x(n,1) - 24y(n,1)$
$$= 3^5 * 324\,[t_{4,n}^2(t_{4,n} - 165) + 1815t_{4,n} - 1331]$$

Solutions for (ii)

$$x(a,b) = 5^5(12\alpha - 96\beta)$$
$$y(a,b) = -5^5(6\alpha + 102\beta)$$
$$z(a,b) = 5^2(a^2 + 11b^2)$$

2.6 Properties

(i) $x(n,1) - 2y(n,1)$
$$= -5^7\{12CP_n^6(6t_{4,n} - 220) + 8712(2t_{3,n} - t_{4,n})\}$$
(ii) $z(2^n - 1, 2^n) = 5^2[12j_{2n} - 3J_{n+1} - 11 - (-1)^{n+1}]$

(iii) $x(n,1) - 48y(n,1) - 5^7 6^2 t_{4,n}[F_{4,n,6} + 3CP_n^6 - 167t_{4,n} + 1815] \equiv 0(mod330)$

Note: 2

Write 1 as

$$1 = \frac{(1+i3\sqrt{11})(1-i3\sqrt{11})}{100} \quad (17)$$

Considering (5), (8), (17) and following the analysis presented in pattern.2, the corresponding integer solutions to (1) are found to be

$$x(a,b) = -10^5(12\alpha + 204\beta)$$
$$y(a,b) = -10^5(44\alpha + 148\beta)$$
$$z(a,b) = 10^2(a^2 + 11b^2)$$

Note:3

One may express 1 in the following two ways :

$$1 = \frac{(7+i5\sqrt{11})(7-i5\sqrt{11})}{324},\ 1 = \frac{(19+i7\sqrt{11})(19-i7\sqrt{11})}{900}$$

Thus, one may obtain two more sets of integer solutions to (1).

3. Conclusion

In this paper, we have made an attempt to determine different patterns of non-zero distinct integer solutions to the non-homogeneous sextic equation with three unknowns $3(x^2 + y^2) - 5xy = 36z^6$. As the sextic equations are rich in variety, one may search for other forms of sextic equation with variables greater than or equal to three and obtain their corresponding properties.

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