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Solution to fractional neutron point kinetic model considering six energy groups using conformable derivatives

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Abstract

In this work the Conformable model, obtained from Fractional Neutron Point Kinetic-model, is extended to consider six groups of delayed neutron precursors. The methodology to determine if the obtained solution is asymptotically stable or unstable of the Conformable model is developed. To show the effect of the functions on the two cases of study (abrupt change of reactivity and reactor start-up), numerical simulations are performed with different anomalous diffusion coefficient values. The results of the Conformable model are equivalent to the results of the classical model. However, the neutron density shows small oscillations around the results of the classical model. The stability analysis is performed through Theorem 1 that establishes the conditions to determine if the solution is asymptotically stable, likewise Theorem 2 is used to determine if the solution is unstable. For both theorems are establish the corresponding inequalities that must be fulfilled.

Keywords: Reactor dynamics; Fractional neutron point kinetic equations; Anomalous diffusion exponent; Conformable derivative; Linear multi-term ordinal differential equation: six energy groups

1. Introduction

Espinosa-Paredes *et al*. (Espinosa-Paredes *et al*., 2011) [17] has derived the fractional neutron point kinetics model (FNPK-model) applying fractional order derivatives to retain the main dynamic characteristics of the neutron motion (Aboanber and Nahla, 2016; Altahhan *et al*., 2016; Espinosa-Paredes and Cázares-Ramírez, 2016; Hamada, 2017) [2, 16, 1, 11]. The physical interpretation of the fractional order derivative is related with non-Fick effects of the neutron diffusion, as consequence of anomalous diffusion phenomena that occur because of the highly heterogeneous configuration in nuclear reactor core (Espinosa-Paredes and Polo-Labarrios, 2012). Even the model considers a relaxation time for neutrons when they start their motion associated with a rapid variation in the neutron flux, due to the fast variation of reactivity. This relaxation time considers that the propagation velocity of neutrons is finite. The FNPK-model constitute a useful tool to provide important information on the dynamics of the reactor and it have been analyzed and applied in different works; e.g., (Cázares-Ramírez *et al*., 2017; Das *et al*., 2013; Nahla and Hemeda, 2017; Nowak *et al*., 2015, 2014a, 2014b; Patra and Saha Ray, 2015; Ray, 2015; Ray and Patra, 2013, 2012; Roul *et al*., 2019; Schramm *et al*., 2013; Vishwesh *et al*., 2017; Vyawahare *et al*., 2018; Vyawahare and Espinosa-Paredes, 2017; Vyawahare and Nataraj, 2013a, 2013b; Vyawaharea and Espinosa-Paredes, 2018) [3, 10].

To solve the FNPK-model, the authors represented their model as a multi-term high-order linear fractional differential equation, thus reduce the problem to a system of ordinary and fractional differential equations, and applied the numerical approximation proposed by Edwards *et al*. (Edwards *et al*., 2002).
The detail description above physical interpretation of the FNPK-model is presented by Espinosa-Paredes *et al*., (Espinosa-Paredes *et al*., 2008) and above its mathematical developed is in Espinosa-Paredes et. al. (Espinosa-Paredes *et al*., 2011) [14], Polo-Labarrios *et al*. (Polo-Labarrios *et al*., 2020), and Espinosa-Paredes *et al*. (Espinosa-Paredes *et al*., 2017) [3].
Recently, Fernández-Anaya *et al*. (Fernández-Anaya *et al*., 2021) investigated the solution of the FNPK-model considering the concept of local fractional order derivatives, and using conformable fractional derivative definition proposed by Khalil *et al*. (Khalil *et al*., 2014). They obtain the Conformable neutron point kinetics model (ConfNPK-model). Their methodology developed consider that there exist an $\alpha-$differentiable functions to able obtain its solution.
The objective of this work is to present the mathematical methodology to perform stability tests to the solution of the ConfNPK-model, considering multigroup of delayed precursors. In detail way considering six groups of delayed neutrons, for two transient cases: in abrupt reactivity changes, and during the dynamic of start-up of a PWR, in both cases two different functions are tested in the ConfNPK-model. The stability analysis is performed through Theorem 1 that establishes the conditions to determine if the solution is asymptotically stable, likewise Theorem 2 is used to determine if the solution is unstable, both theorems given by Chen and Yang (Chen and Yang, 2016). The neutron density obtained from the ConfNPK-model and the Classical neutron point kinetic model (Classical-model) is compared to show the effect of the proposed functions and the anomalous diffusion coefficient value, for two transient cases. The comparison shows that ConfNPK-model results are agree with the Classical-model, but oscillatory behavior is observed, it is depending in both function and anomalous diffusion value.

2. ConfNPK-model for six group of delayed precursor neutron

Espinosa-Paredes *et al*. (Espinosa-Paredes *et al*., 2011) [14] modified the Classical-model based on the Cattaneo´s laws. They have extended the application of these laws by applying a fractional constitutive law. The goal of this model is describing the diffusion processes that do not follow the Fick´s diffusion law observed in many natural systems. Their model obtained is knows as Fractional Neutron Point Kinetic-model (Espinosa-Paredes *et al*., 2011, 2008; Hamada, 2017; Polo-Labarrios *et al*., 2020) [16, 14]. Recently, Fernandez-Anaya *et al*. (Fernández-Anaya *et al*., 2021) applied the conformable definition to obtain the following model:

$$\tau^{\alpha} f(t)^{1-\alpha}\frac{d^2N(t)}{dt^2}+\tau^{\alpha} f(t)^{1-\alpha}\left(\frac{P_{NL}\left[1-\rho(t)\right]-1+\beta}{\Lambda}\right)\frac{dN(t)}{dt}$$
$$-\tau^{\alpha}\frac{f(t)^{1-\alpha}P_{NL}N(t)}{\Lambda}\frac{d\rho(t)}{dt}+\frac{dN(t)}{dt}=\frac{\rho(t)-\beta}{\Lambda}N(t) \qquad 0<\alpha<1 \tag{1}$$
$$+\tau^{\alpha} f(t)^{1-\alpha}\sum_{i=1}^{m}\lambda_i\frac{dC_i(t)}{dt}+\sum_{i=1}^{m}\lambda_i C_i(t)$$

where t is the time, $f(t)$ is an $\alpha-$differentiable functions used to obtain a stable solution, $N(t)$ is the neutron density, P_{NL} is the neutron non-leakage probability, $\rho(t)$ is the reactivity, Λ is one generation average lifetime of instantaneous neutron, λ is the decay constant of delayed neutrons precursors, $C(t)$ is the delayed neutrons precursors density, β is the total fraction of delayed neutrons precursors, defined as

$$\beta=\sum_{i=1}^{m}\beta_i \tag{2}$$

α is the order of the differential operator known as the anomalous diffusion coefficient (for sub-diffusion process: $0<\alpha<1$; while for super-diffusion process: $1<\alpha<2$). τ^{α} is the anomalous relaxation time, and it is defined as (Espinosa-Paredes *et al.*, 2008):

$$\tau=\frac{1}{\upsilon\Sigma_{tr}}=\frac{3D}{\upsilon} \tag{3}$$

Where υ is the neutron velocity, Σ_{tr} is the transport cross-section, and D is the diffusion coefficient. Other way, differential equation of delayed neutron precursors is:

$$\frac{dC_i(t)}{dt}=\frac{\beta_i}{\Lambda}N(t)-\lambda_i C_i(t) \qquad (4)$$

The Eqs. (1) and (4) are a couple of stiff second order nonlinear ordinary integer order differential equations. The goal of that model is analyzing the effects of sub-diffusion processes $(0<\alpha<1)$ due the anomalous diffusion exponent and the relaxation time, applying conformable definition using different function. The Classical-model of the point kinetic equations is obtained directly from Eq. (1) when $\tau=0$.

3. ConfNPK-model stability behavior for six group of delayed precursor neutron

This section presents the development of the ConfNPK-model considering six groups of delayed precursor neutrons. Rewriting the Eqs. (1) and (4) in terms of a systems of differential equations of third order

$$\begin{aligned}
\frac{dN(t)}{dt}&=M(t)\\
\frac{dM}{dt}&=-\left(\frac{P_{NL}\left[1-\rho(t)\right]-1+\beta}{\Lambda}\right)M(t)+\frac{P_{NL}}{\Lambda}N(t)\frac{d\rho(t)}{dt}-\tau^{\alpha}f(t)^{\alpha-1}M(t)\\
&\quad+\tau^{-\alpha}f(t)^{\alpha-1}\left(\frac{\rho(t)-\beta}{\Lambda}\right)N(t)+\sum_{i=1}^{m}\lambda_i\left[\frac{\beta_i}{\Lambda}N(t)-\lambda_i C_i(t)\right]+\sum_{i=1}^{m}\tau^{\alpha}f(t)^{\alpha-1}\lambda_i C_i(t)\\
\frac{dC_i(t)}{dt}&=\frac{\beta_i}{\Lambda}N(t)-\lambda_i C_i(t)
\end{aligned} \qquad (5)$$

Simply the system (5)

$$\begin{aligned}
\frac{dN(t)}{dt}&=M(t)\\
\frac{dM}{dt}&=-\left(\frac{P_{NL}\left[1-\rho(t)\right]-1+\beta}{\Lambda}+\tau^{-\alpha}f(t)^{\alpha-1}\right)M(t)\\
&\quad+\left[\frac{P_{NL}}{\Lambda}\frac{d\rho(t)}{dt}+\tau^{-\alpha}f(t)^{\alpha-1}\left(\frac{\rho(t)-\beta}{\Lambda}\right)+\sum_{i=1}^{m}\frac{\lambda_i\beta_i}{\Lambda}\right]N(t)\\
&\quad+\sum_{i=1}^{m}\left[\tau^{-\alpha}f(t)^{\alpha-1}\lambda_i-\lambda_i^2\right]C_i(t)\\
\frac{dC_i(t)}{dt}&=\frac{\beta_i}{\Lambda}N(t)-\lambda_i C_i(t)
\end{aligned} \qquad (6)$$

The system (6) has the following representation for six groups:

$$\begin{pmatrix} \frac{dM(t)}{dt} \\ \frac{dC_1(t)}{dt} \\ \frac{dC_2(t)}{dt} \\ \frac{dC_3(t)}{dt} \\ \frac{dC_4(t)}{dt} \\ \frac{dC_5(t)}{dt} \\ \frac{dC_6(t)}{dt} \\ \frac{dN(t)}{dt} \end{pmatrix} = \begin{pmatrix} M_1(t) & M_{21}(t) & M_{22}(t) & M_{23}(t) & M_{24}(t) & M_{25}(t) & M_{26}(t) & M_3(t) \\ 0 & -\lambda_1 & 0 & 0 & 0 & 0 & 0 & \frac{\beta_1}{\Lambda} \\ 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 & \frac{\beta_2}{\Lambda} \\ 0 & 0 & 0 & -\lambda_3 & 0 & 0 & 0 & \frac{\beta_3}{\Lambda} \\ 0 & 0 & 0 & 0 & -\lambda_4 & 0 & 0 & \frac{\beta_4}{\Lambda} \\ 0 & 0 & 0 & 0 & 0 & -\lambda_5 & 0 & \frac{\beta_5}{\Lambda} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_6 & \frac{\beta_6}{\Lambda} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} M(t) \\ C_1(t) \\ C_2(t) \\ C_3(t) \\ C_4(t) \\ C_5(t) \\ C_6(t) \\ N(t) \end{pmatrix} \quad (7)$$

Where

$$\begin{aligned} M_1(t) &= -\frac{P_{NL}\left[1-\rho(t)\right]-1+\beta}{\Lambda} - \tau^{-\alpha} f(t)^{\alpha-1} \\ M_{2i}(t) &= \tau^{-\alpha} f(t)^{\alpha-1}\lambda_i - \lambda_i^2 \\ M_3(t) &= \frac{P_{NL}}{\Lambda}\frac{d\rho(t)}{dt} + \sum_{i=1}^{m}\frac{\lambda_i\beta_i}{\Lambda} + \tau^{-\alpha} f(t)^{\alpha-1}\left(\frac{\rho(t)-\beta}{\Lambda}\right) \end{aligned} \quad (8)$$

Now, this work considers an initially critical reactor that has been operating at steady state, at $t=0$ a constant change of reactivity is inserted, it is:

$$\rho(t) = \begin{cases} 0, & t<0 \\ \left(p_n t^n + \cdots + p_1 t + p_0\right)e^{-\zeta t}, & t \ge 0,\ \zeta \ge 0,\ p_i \ge 0, \quad \text{for } i = 0,\ldots,n \end{cases} \quad (9)$$

where p_i are the coefficients of the polynomial with adequate units and ζ represents a frequency. Substituting Eq. (9) into (8),

$$\begin{aligned} M_1(t) &= -\frac{P_{NL}\left[1-\left(p_n t^n + \cdots + p_1 t + p_0\right)e^{-\zeta t}\right]-1+\beta}{\Lambda} - \tau^{-\alpha} f(t)^{\alpha-1} \\ M_{2i}(t) &= \lambda_i \tau^{-\alpha} f(t)^{\alpha-1} - \lambda_i^2, \qquad \text{for} \quad i = 1,\ldots,6 \\ M_3(t) &= \frac{P_{NL}}{\Lambda}\left(\left(\left((n-1)p_n t^{n-1} + \cdots + p_1\right)e^{-\zeta t}\right) - \left(p_n t^n + \cdots + p_1 t + p_0\right)\frac{e^{-\zeta t}}{\zeta}\right) + \\ &\quad \sum_{i=1}^{6}\frac{\lambda_i\beta_i}{\Lambda} + \tau^{-\alpha} f(t)^{\alpha-1}\left(\frac{\left(p_n t^n + \cdots + p_1 t + p_0\right)e^{-\zeta t} - \beta}{\Lambda}\right) \end{aligned} \quad (10)$$

Simply the equations system (10),

$$
\begin{aligned}
&M_1(t) = -\frac{P_{NL}\left[1-\left(p_n t^n + \cdots + p_1 t + p_0\right)e^{-\zeta t}\right]-1+\beta}{\Lambda} - \tau^{-\alpha} f(t)^{\alpha-1} \\
&M_{2i}(t) = \lambda_i \tau^{-\alpha} f(t)^{\alpha-1} - \lambda_i^2, \qquad \text{for} \quad i = 1,\ldots,6 \\
&M_3(t) = \frac{P_{NL}}{\Lambda}\left(-\frac{p_n}{\zeta}t^n + \left((np_n - \frac{p_{n-1}}{\zeta}\right)t^{n-1} + \cdots + \left(2p_2 - \frac{p_1}{\zeta}\right)t + p_1 - \frac{p_0}{\zeta}\right)e^{-\zeta t} + \\
&\qquad \sum_{i=1}^{6}\frac{\lambda_i \beta_i}{\Lambda} + \tau^{-\alpha} f(t)^{\alpha-1}\left(\frac{\left(p_n t^n + \cdots + p_1 t + p_0\right)e^{-\zeta t} - \beta}{\Lambda}\right)
\end{aligned} \tag{11}
$$

When the concept of local fractional derivatives is used, the properties of ordinary derivatives are keeping, and thus obtaining the ConfNPK-model. Whose solution needs an $\alpha-$differentiable functions. Is needed guaranteed the existences of that function. In this section a methodology to determinates the constraint conditions that the function must fulfill.
Now, based on **Theorem 2** of Chen and Yang (Chen and Yang, 2016), sufficient condition is established to have asymptotic stability, with $f(t)$ non-periodic. The condition is as follows

$$
\int_{t_0}^{\infty}\sum_{i=0}^{8}\lambda_{\max}\left(B_i + B_i^T\right)g_i(t)dt \to -\infty \tag{12}
$$

Where $\lambda_{\max}$ is the maximum value of the eigenvalue, and $g_i(t)$ are given by

$$
\begin{aligned}
&g_0(t) = 1 \\
&g_1(t) = M_1(t) \\
&g_2(t) = M_{21}(t) \\
&g_3(t) = M_{22}(t) \\
&g_4(t) = M_{23}(t) \\
&g_5(t) = M_{24}(t) \\
&g_6(t) = M_{25}(t) \\
&g_7(t) = M_{26}(t) \\
&g_8(t) = M_3(t)
\end{aligned} \tag{13}
$$

Notice that:

$$\begin{pmatrix} \frac{dM(t)}{dt} \\ \frac{dC_1(t)}{dt} \\ \frac{dC_2(t)}{dt} \\ \frac{dC_3(t)}{dt} \\ \frac{dC_4(t)}{dt} \\ \frac{dC_5(t)}{dt} \\ \frac{dC_6(t)}{dt} \\ \frac{dN(t)}{dt} \end{pmatrix} = \begin{pmatrix} M_1(t) & M_{21}(t) & M_{22}(t) & M_{23}(t) & M_{24}(t) & M_{25}(t) & M_{26}(t) & M_3(t) \\ 0 & -\lambda_1 & 0 & 0 & 0 & 0 & 0 & \frac{\beta_1}{\Lambda} \\ 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 & \frac{\beta_2}{\Lambda} \\ 0 & 0 & 0 & -\lambda_3 & 0 & 0 & 0 & \frac{\beta_3}{\Lambda} \\ 0 & 0 & 0 & 0 & -\lambda_4 & 0 & 0 & \frac{\beta_4}{\Lambda} \\ 0 & 0 & 0 & 0 & 0 & -\lambda_5 & 0 & \frac{\beta_5}{\Lambda} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_6 & \frac{\beta_6}{\Lambda} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} M(t) \\ C_1(t) \\ C_2(t) \\ C_3(t) \\ C_4(t) \\ C_5(t) \\ C_6(t) \\ N(t) \end{pmatrix}$$

$$= \frac{d\vec{C}(t)}{dt} = A(t)\vec{C}(t) \tag{14}$$

To obtain the result of the equation system (14), first convert the system matrix function $A(t)$ into the following equivalent form:

$$A(t) = \sum_{i=0}^{8} B_i g_i(t) \tag{15}$$

Where $g_i(t)$ for $i = 0, \ldots, 8$ are functions given by Eq. (13), they satisfying $g_i(t) \geq 0$ for $t \geq t_0$. And B_i are constant matrices given by:

$$B_0 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\lambda_1 & 0 & 0 & 0 & 0 & 0 & \frac{\beta_1}{\Lambda} \\ 0 & 0 & -\lambda_2 & 0 & 0 & 0 & 0 & \frac{\beta_2}{\Lambda} \\ 0 & 0 & 0 & -\lambda_3 & 0 & 0 & 0 & \frac{\beta_3}{\Lambda} \\ 0 & 0 & 0 & 0 & -\lambda_4 & 0 & 0 & \frac{\beta_4}{\Lambda} \\ 0 & 0 & 0 & 0 & 0 & -\lambda_5 & 0 & \frac{\beta_5}{\Lambda} \\ 0 & 0 & 0 & 0 & 0 & 0 & -\lambda_6 & \frac{\beta_6}{\Lambda} \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{16}$$

For the matrix $B_{8\times 8}$ with elements $b_{1,j} = 1$ for $j = 1 \ldots 8$, for example:

$$B_{1,j}=\begin{pmatrix} 0 & 0 & b_{1,j} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \quad \text{for } j=3 \tag{17}$$

Thus, to satisfy Eq. (12) the maximum eigenvalues are:

1. $\mu_0=\lambda_{\max}\left(B_0+B_0^T\right)=\sigma$,
2. $\mu_1=\lambda_{\max}\left(B_1+B_1^T\right)=2$,
3. $\mu_i=\lambda_{\max}\left(B_i+B_i^T\right)=1, \quad \text{for} \quad i=2,\ldots,8$

Where σ depends on the numerical values λ and β/Λ.

3.1. Asymptotically stable of ConfNPK-model

This section presents the methodology to determine whether ConfNPK-model solution is stable from the proposed function. For solve the equivalent form of $A(t)$ in Eq. (15), the **Theorem 1** proposed by Chen and Yang (Chen and Yang, 2016) is used. It says that the system of Eqs. (14) with equivalent form of $A(t)$ given in Eq. (15), is asymptotically stable if

$$\int_{t_0}^{+\infty}\left[\mu_0+\mu_1 g_1(t)+\cdots+\mu_8 g_8(t)\right]dt=-\infty \tag{18}$$

where $\mu_j=\lambda_{\max}\left(B_j^*\right)$, $B_j^*=B_j(t)^T+B_j(t)$, for $j=0,1,\ldots,8$ and with $g_0(t)=1$, $g_1(t)=M_1(t)$, $g_{i+1}(t)=M_{2i}(t)$, for $i=1,\ldots,6$ and $g_8(t)=M_8(t)$.

The following condition guarantees asymptotic stability

$$\int_{t_0}^{\infty}\left[\begin{array}{l} \sigma-2\left(\dfrac{P_{NL}\left[1-\left(p_n t^n+\cdots+p_1 t+p_0\right)e^{-\zeta t}\right]-1+\beta}{\Lambda}+\tau^{-\alpha}f(t)^{\alpha-1}\right) \\ +\displaystyle\sum_{i=1}^{6}\left(\lambda_i\tau^{-\alpha}f(t)^{\alpha-1}-\lambda_i^2\right) \\ +\dfrac{P_{NL}}{\Lambda}\left(-\dfrac{p_n}{\zeta}t^n+\left(np_n-\dfrac{p_{n-1}}{\zeta}\right)t^{n-1}+\cdots+\left(2p_2-\dfrac{p_1}{\zeta}\right)t+p_1-\dfrac{p_0}{\zeta}\right)e^{-\zeta t} \\ +\displaystyle\sum_{i=1}^{6}\dfrac{\lambda_i\beta_i}{\Lambda}+\tau^{-\alpha}f(t)^{\alpha-1}\left(\dfrac{\left(p_n t^n+\cdots+p_1 t+p_0\right)e^{-\zeta t}-\beta}{\Lambda}\right) \end{array}\right]dt=-\infty \tag{19}$$

From Eq. (19) are taken the following definitions for a subsequent analysis of the Eq (14),

$$P(t) = \tau^{-\alpha}\left[-2+\sum_{i=1}^{6}\lambda_i + \left(\frac{\left(p_n t^n + \text{L} + p_1 t + p_0\right)e^{-\zeta t} - \beta}{\Lambda}\right)\right], \text{ and} \quad (20)$$

$$T(t) = \sigma - 2\left(\frac{P_{NL}\left[1-\left(p_n t^n + \cdots + p_1 t + p_0\right)e^{-\zeta t}\right] - 1 + \beta}{\Lambda}\right) + \frac{P_{NL}}{\Lambda}\left(-\frac{p_n}{\zeta}t^n + \left(np_n - \frac{p_{n-1}}{\zeta}\right)t^{n-1} + \cdots + \left(2p_2 - \frac{p_1}{\zeta}\right)t + p_1 - \frac{p_0}{\zeta}\right)e^{-\zeta t} - \sum_{i=1}^{6}\lambda_i^2 + \sum_{i=1}^{6}\frac{\lambda_i\beta_i}{\Lambda} \quad (21)$$

In this way Eq. (19) is divided into two integrals: one of them is an integral that factorizes the terms multiplied by $\tau^{-\alpha} f(t)^{\alpha-1}$, and another integral that groups the rest of the terms. Then the Eq. (19) remains as:

$$\tau^{-\alpha}\int_{t_0}^{\infty} P(t) f(t)^{\alpha-1}\,dt + \int_{t_0}^{\infty} T(t)dt = -\infty \quad (22)$$

Some possibilities to fulfill the condition given by Eq. (22) are presented below:
1. If the following condition is satisfying:

$$\sigma - 2\left(\frac{P_{NL} - 1 + \beta}{\Lambda}\right) - \sum_{i=1}^{6}\lambda_i^2 + \sum_{i=1}^{6}\frac{\lambda_i\beta_i}{\Lambda} < 0 \quad (23)$$

It is possible to satisfy the condition (22) taking $f(t) = \tau e^{-at}, 0 < a, 0 < \zeta,$ or $f(t) = \tau^{1+\alpha}t^{-\alpha}$, with $a > 1$ and $\alpha \in (0,1)$, when $t \to \infty$. Note that condition (23) does not depend on the reactivity $\rho(t)$.

In the case when $\rho(t)$ is a constant p_0 the inequality (23) is now

$$\sigma - 2\left(\frac{P_{NL}(1-p_0) - 1 + \beta}{\Lambda}\right) - \sum_{i=1}^{6}\lambda_i^2 + \sum_{i=1}^{6}\frac{\lambda_i\beta_i}{\Lambda} < 0 \quad (24)$$

Where $\zeta = 0$ and the same previous definitions (20) and (21).
2. In the case when

$$\sigma - 2\left(\frac{P_{NL} - 1 + \beta}{\Lambda}\right) - \sum_{i=1}^{6}\lambda_i^2 + \sum_{i=1}^{6}\frac{\lambda_i\beta_i}{\Lambda} > 0 \quad (25)$$

It is possible to satisfy the condition (22) taking $f(t) = \tau e^{at}, 0 < a < \zeta,$ or $f(t) = \tau^{1-\alpha}t^{\alpha}$, with $a > 0$ and $\alpha \in (0,1)$ and $-2+\sum_{i=1}^{6}\lambda_i - \frac{\beta}{\Lambda} < 0$, when $t \to \infty$. Note that condition (25) does not depend on the reactivity $\rho(t)$.

In the case when $\rho(t)$ is a constant p_0 the inequality (25) is now

$$\sigma - 2\left(\frac{P_{NL}\left(1-p_0\right)-1+\beta}{\Lambda}\right) - \sum_{i=1}^{6}\lambda_i^2 + \sum_{i=1}^{6}\frac{\lambda_i\beta_i}{\Lambda} > 0 \tag{26}$$

Where $\zeta = 0$, with $-2+\sum_{i=1}^{6}\lambda_i - \left(\frac{p_0-\beta}{\Lambda}\right) < 0$ the condition when $\zeta = 0$ and the same previous definitions (20) and (21).

3.2. Instability of ConfNPK-model

On the other hand, to know if the system is Unstable, the following condition established by Wu's **Theorem 2** (Wu, 1982) guarantees instability for any $t > t_0$

$$\int_{t_0}^{\infty} \mathrm{tr}\, A(t)\, dt \to \infty \tag{27}$$

The above condition (27) is true, if the following condition is satisfied.

$$-\int_{t_0}^{\infty}\left(\frac{P_{NL}\left[1-\left(p_n t^n + \cdots + p_1 t + p_0\right)e^{-\zeta t}\right]-1+\beta}{\Lambda} + \tau^{-\alpha} f(t)^{\alpha-1} + \sum_{i=1}^{6}\lambda_i\right)dt \to \infty \tag{28}$$

It is possible to meet this condition (28) by taking:

$$\int_{t_0}^{\infty} f(t)^{\alpha-1}\, dt \to 0 \text{ and } Z = \frac{P_{NL} - 1 + \beta}{\Lambda} + \sum_{i=1}^{6}\lambda_i < 0 \tag{29}$$

Which is achieved for example with $f(t) = \tau e^{at}$, with $\zeta > 0$, $a < 0$. Also taking $f(t) = \tau^{1+\alpha} t^{-a}$ and with $a > 1$, $\alpha \in (0,1)$, when $t \to \infty$. Note that condition (29) does not depend on the reactivity $\rho(t)$.

In the case when $\rho(t)$ is a constant p_0 the inequality (29) is now

$$\int_{t_0}^{\infty} f(t)^{\alpha-1}\, dt \to 0 \text{ and } Z = \frac{P_{NL}\left(1-p_0\right)-1+\beta}{\Lambda} + \sum_{i=1}^{6}\lambda_i < 0 \tag{30}$$

Where $\zeta = 0$ and the same previous condition (30). It is clear, that more cases are possible.

4. Numerical simulation considering six-groups delayed neutron

The behavior of the neutron density is studied in two cases: abrupt change of reactivity and start-up dynamics of a PWR reactor, in both cases the reactivity insertion is constant. To determine if the solution obtained with the ConfNPK-model is stable, the methodology developed in this work is applied. Two functions are selected to solve this model:

Function i) $f(t) = \tau e^{at}$, (31)

Function ii) $f(t) = \tau^{1-a} t^{a}$, (32)

Where the system (6) is solved using the finite difference numerical method in implicit scheme (Chapra and Canale, 2009). The physic parameters used to numerical simulation are presented in **Table 1**.

Table 1: Physical parameters (Hamada, 2017) [16].

Parameter	Value	Units
D	0.0921	$cm^2 s^{-1}$
υ	2200	m/s
$\tau = 3D/\upsilon$	1.256×10^{-4}	s
α	$0<\alpha<1$	—
P_{NL}	0.975	—

4.1. Abrupt reactivity changes

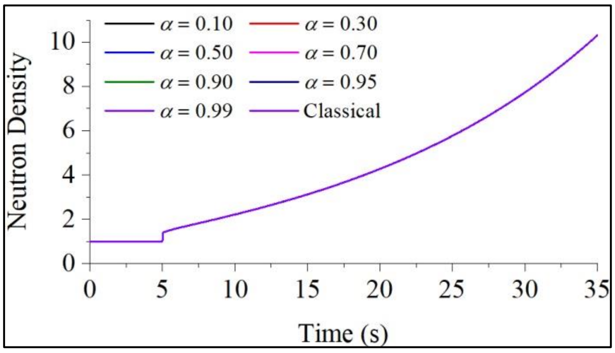
Considers the case of step reactivity value at $t=0$ a constant change. The experiments consider the insertion of reactivity step $\rho = 0.002$ to perturb the system. The neutron density behavior is studied by using the values of the anomalous diffusion coefficient $\alpha =$ 0.99, 0.98, 0.97, 0.95, 0.93, 0.91, 0.90, $\Lambda = 0.00005\,s$ and for the decay constant of delayed neutron precursor λ_i and the total fraction of delayed neutron β_i used in this work are presented in **Table 2**.

Table 2: Neutron delayed fractions and decay constants, for six precursor group parameters (Chao and Attard, 1985; Hamada, 2015; Nobrega, 1971) [17].

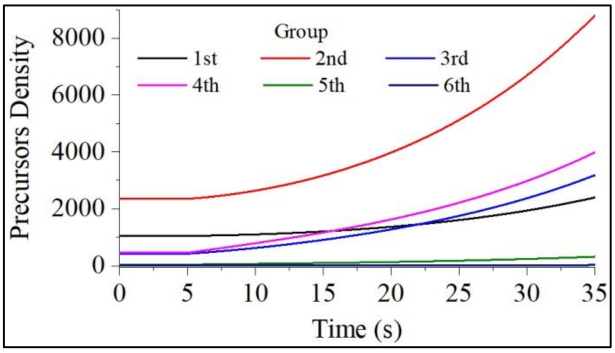
i	λ_i (s^{-1})	β_i
1	0.0124	0.00021
2	0.0305	0.00141
3	0.1110	0.00127
4	0.3010	0.00255
5	1.1300	0.00074
6	3.0000	0.00027

The initial equilibrium conditions for neutron density is $n_0 = 1$ and for six-group of delayed neutrons density are $C_{i,0} = \beta_i n_0 / \Lambda\lambda_i$.

With reactivity value $\rho = 0.002$, the **Figure 1** and **2** present the results using the functions $f(t) = \tau e^{at}$ and $f(t) = \tau^{1-a} t^a$, repectively, both with $a = 1$. It can see, all cases the behavior of neutron density diverge. In both cases, it can be seen that the value of the anomalous diffusion coefficient has no effect on the behavior of the precursors delayed neutron density (**Figures 1b** and **2b**).

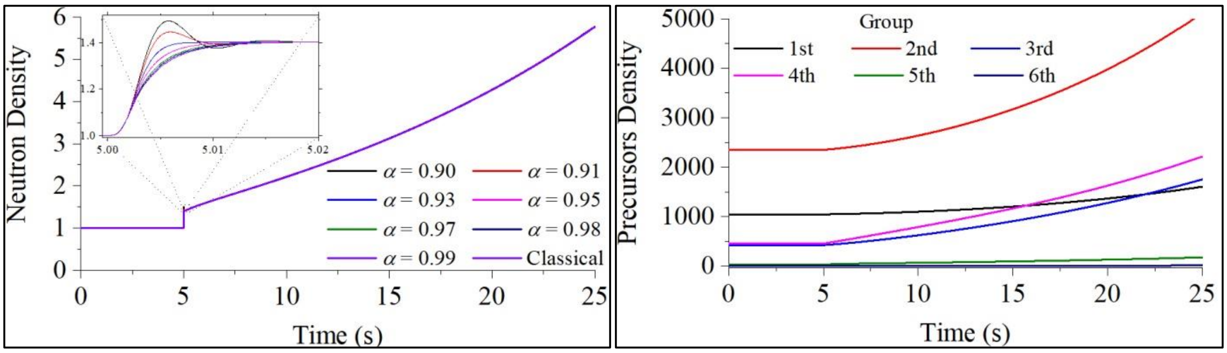


A



B

Fig 1: a) Neutron density and b) Precursor density behavior; both using ConfNPK-model for using six-groups of delayed precursors density. For different anomalous diffusion constant $\alpha =$ 0.10 to 0.99, with constant reactivity value $\rho = 0.002$ and $f(t) = \tau e^{at}$ with $a = 1$.



A B

Fig 2: a) Neutron density and b) Precursor density behavior; both using ConfNPK-model for using six-groups of delayed precursors density. For different anomalous diffusion constant $\alpha = 0.90$ to 0.99, with constant reactivity value $\rho = 0.002$ and $f(t) = \tau^{1-a} t^{a}$ with $a = 1$.

Now, the **Figure 2a** shows a zoom to see the neutron density behavior around 5 s using $f(t) = \tau^{1-a} t^{a}$. As the value of the anomalous diffusion coefficient decreases, the curves move away from the solution of the Classical-model.

4.1.1. Analysis of stability behavior based in the results

Based on **Theorem 1** and the **Tables 1** and **2**, we cannot conclude an asymptotic stable dynamic. The Eq. (26) is positive

$$\sigma - 2\left(\frac{P_{NL}(1-p_0)-1+\beta}{\Lambda}\right) - \sum_{i=1}^{6}\lambda_i^2 + \sum_{i=1}^{6}\frac{\lambda_i\beta_i}{\Lambda} \approx 2536.4 > 0 \quad (33)$$

But the following inequality, makes impossible conclude asymptotic stability of the **Theorem 1**

$$-2 + \sum_{i=1}^{6}\lambda_i - \left(\frac{p_0-\beta}{\Lambda}\right) = 102.58 > 0 \quad (34)$$

Similarly, of the **Tables 2**, with values given in **Tables 3, 4**, presented by the authors Kinard and Allen (Kinard and Allen, 2004) and Quintero-Leyva (Quintero-Leyva, 2008), respectively, we cannot conclude an asymptotic stable dynamic. Notice that the last result is independent of the value of α.

Table 3: Neutron delayed fractions and decay constants, for six precursor group parameters (Kinard and Allen, 2004).

i	***λi* (s⁻¹)**	***βi***
1	0.0127	0.000266
2	0.0317	0.001491
3	0.1150	0.001316
4	0.3110	0.002849
5	1.400	0.000896
6	3.8700	0.000182

Table 4: Neutron delayed fractions and decay constants, for six precursor group parameters (Quintero-Leyva, 2008).

i	***λi* (s⁻¹)**	***βi***
1	0.0124	0.000215
2	0.0305	0.001424
3	0.1110	0.001274
4	0.3010	0.002568
5	1.1400	0.000748
6	3.0100	0.000273

Now, based on **Theorem 2** and with the values present in **Tables 1** to **4**, in the following three cases we have an unstable dynamic.

Taking $\rho = 0.002$ and the **Table 1** with either $f(t) = \tau e^{-t}$ or $f(t) = \tau^{6} t^{-5}$ with $\alpha = 0.5$.

Using the **Table 2** and inequality (30), we obtain

$$Z = \frac{P_{NL}(1-p_0)-1+\beta}{\Lambda} + \sum_{i=1}^{6} \lambda_i = 890.51 > 0 \tag{35}$$

Using the **Table 3** and inequality of Eq. (30), we obtain

$$Z = \frac{P_{NL}(1-p_0)-1+\beta}{\Lambda} + \sum_{i=1}^{6} \lambda_i = 876.09 > 0 \tag{36}$$

Using the **Table 4** and inequality of Eq. (30), we obtain

$$Z = \frac{P_{NL}(1-p_0)-1+\beta}{\Lambda} + \sum_{i=1}^{6} \lambda_i = 888.93 > 0 \tag{37}$$

4.2. Dynamics of start-up of PWR nuclear reactor

The case of neutron multiplication during start-up of PWR, is a process where is assumed that during the i -th step of control rod withdrawal the way of reactivity insertion is step, the neutron source strength was defines as a constant in terms of a known initial stable sub-criticality and the neutron signal from a steady state condition (Duderstadt and Hamilton, 1976; Espinosa-Paredes and Polo-Labarrios, 2012; Li *et al*., 2010).

The representative FNPK-model for this case correspond with Eq. (1), with six-group of neutrons delayed precursors and considering an external source *q*, is represented by:

$$\begin{aligned} &\tau^{\alpha} f(t)^{1-\alpha} \frac{d^2 N(t)}{dt^2} + \tau^{\alpha} f(t)^{1-\alpha} \left(\frac{P_{NL}\left[1-\rho(t)\right]-1+\beta}{\Lambda} \right) \frac{dN(t)}{dt} \\ &\quad -\tau^{\alpha} \frac{f(t)^{1-\alpha} P_{NL} N(t)}{\Lambda} \frac{d\rho(t)}{dt} + \frac{dN(t)}{dt} = \frac{\rho(t)-\beta}{\Lambda} N(t) \\ &\quad +\tau^{\alpha} f(t)^{1-\alpha} \sum_{i=1}^{6} \lambda_i \frac{dC_i(t)}{dt} + \sum_{i=1}^{6} \lambda_i C_i(t) + q \end{aligned} \tag{38}$$

where q is the strength neutron source density emitted per second. When the rector power is reaches steady-state, all time derivatives are equal to zero, so ρ_0 is a subcritical reactivity initial value.

The nuclear parameters used in this case the values from **Table 3** are used for λ and β, $\Lambda = 10^{-4}$ s and $q = 1\times10^{8}$ neutrons/m^{3}s, they correspond to a PWR with ^{235}U as fissile material (Li *et al*., 2010). It is assumed that, before the sudden change in reactivity occurred at $t_1 = 0$, the initial count of neutron detector is equal to the initial value of neutron density (n_0). Then the initial sub-critical value is $\rho_0 = -100\,\text{mk}$, it is used to obtain the initial value to density and delayed neutron precursor density from $\rho_0 = -q\Lambda/n_0$ and $C_0 = \beta n_0/\lambda\Lambda$, respectively. When withdraw control rod at different time the sub-critical values added are presented in **Table 5**.

Table 5: Sub-critical values added when withdraw the control rod.

Time (s)	Reactivity (mk)
$t_1 = 0$	$\rho_1 = -10.00$
$t_2 = 100$	$\rho_2 = -5.00$
$t_3 = 200$	$\rho_3 = -2.50$
$t_4 = 305$	$\rho_4 = -1.25$
$t_5 = 425$	$\rho_5 = -1.00$
$t_6 = 605$	$\rho_6 = -0.75$

The results using the Eq. (38) are shown in **Figures 3**, it presented both neutron density and delayed precursors density behaviors, using different anomalous diffusion coefficient values ($\alpha =$ 0.795 to 0.990) and with $f(t) = \tau^{1+a} t^{-a}$ with $a = 1$: also, different zooms to show the behavior of the neutron density obtained when reactivity perturbations is inserted are presented. In this case the first function, $f(t) = \tau e^{-at}$, have not any effect on neutron density behavior during start-up nuclear reactor, even neither the anomalous diffusion coefficient value. It means, the function selected neutralized the anomalous diffusion coefficient effect on neutron density behavior.

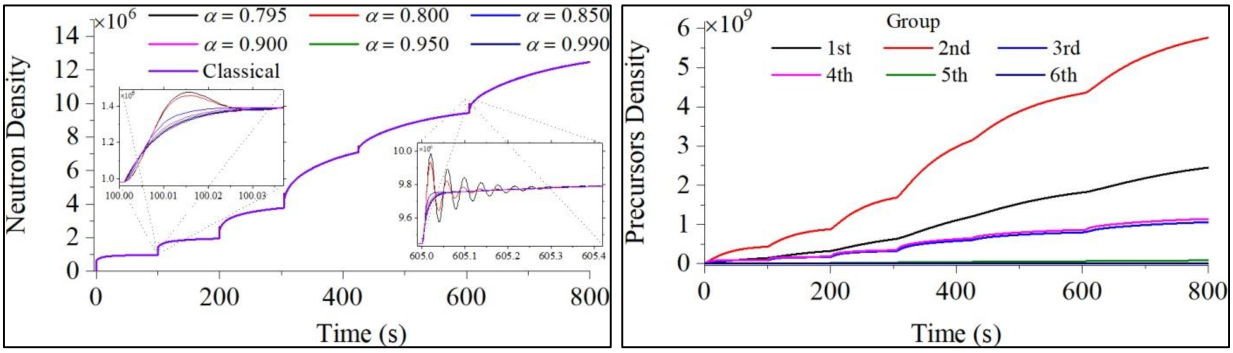


Fig 3: Neutron density behavior after inserting step reactivity into a sub-critical core. Using ConfNPK-model for six grups of delayed precursor neutrons, for different anomalous diffusion coefficient values $\alpha =$ 0.795 to 0.990 and $f(t) = \tau^{1+a} t^{-a}$ with $a = 1$: a) Neutron density behavior in full range of the simulation time: its shown a zoom in the simulation range from 100 s to 100.04 s; and a zoom in the simulation range from 605 s to 605.4 s. b) Delayed precursors density behavior in full range of the simulation time, for all anomalous diffusion coefficient values.

4.2.1. Analysis of stability behavior based in the results

To determinate the dynamics of start-up of PWR nuclear reactor for stability behavior based in the results, in the **Table 6** we use **Theorem 1** criteria 1) inequality (24), for the first three columns. In the fourth column we use criterion 2) of **Theorem 1** inequality (26).

Table 6: Stability for function $f(t) = \tau e^{-at}$ with the values of the **Tables 2** to **4**.

Reactivity (mk)	Table 2	Table 3	Table 4
ρ_1 = -10.00	Stable	Stable	Stable
ρ_2 = -5.00	Stable	Stable	Stable
ρ_3 = -2.50	Stable	Stable	Stable
ρ_4 = -1.25	Stable	Stable	Stable
ρ_5 = -1.00	Stable	Stable	Stable
ρ_6 = -0.75	Stable	Stable	Stable

It can see that the neutron density behavior calculated by ConfNPK-model with six-groups of delayed neutrons are agrees with Classical-models results. The results shown, **Figure 3**, that at time of the sub-critical values is added, when withdraw the control rod, the neutron density present a oscillatory behavior, whose duration increase in each time. Also, the oscillation increases when the anomalous diffusion coefficient decreases. For anomalous diffusion coefficient value less than 0.795, the ConfNPK-model solution is unstable.

Recently, Polo-Labarriosa et. al. (Polo-Labarrios *et al*., 2022) and Fernández-Anaya et. al. (Fernández-Anaya *et al*., 2021) developed both analytical–numerical solution and ConfNPK-model, respectively. Their results, obtained with other methods, are agreed with obtained in this work. In the first work the authors found the numerical simulations describe inertia effects observed as a growth in neutron density up to reaching a peak and then a gradual decrease followed by a series of oscillations until reaching a steady state. This behavior is accentuated as the fractional order decreases. While, in second work the authors found that in ConfNPK-model results an oscillatory behavior is observed, the authors mention that this depending in both $\alpha-$ differentiable function and anomalous diffusion value used in the solution.

5. Conclusion

In this work the conformable model, obtained from the fractional model considering the concept of fractional order local derivative, was extended to consider groups of delayed neutrons, specifically six. That model was called conformable neutron point kinetic model, i.e., ConfNPK-model.

A methodology to determine the stability of the solution obtained from the ConfNPK-model was developed and tested for two transient cases of study: abrupt change of reactivity and start-up dynamics of a PWR, where the inserting reactivity is a linear through the reactor start-up time and a constant after that. In both cases two different functions were used and the value of the anomalous diffusion coefficient was varied. The stability analysis shows that with both functions the start-up dynamics of the PWR is stable.

Additionally, a comparison of the ConfNPK-model and Classical-model results is performed to show the effect of the anomalous diffusion coefficient. The comparison shows that the results of both models are equivalent, except in small regions where the ConfNPK-model results show small oscillations with respect to the Classical-model results. These oscillations increase in amplitude and duration as the value of the anomalous diffusion coefficient decreases.

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