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The method of separation of variables in the resolution of PDEs

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Abstract
It is well known that there is no standard method in solving second-order partial differential equations. Thus, in this article we present a particular method: the separation of variables. As the name suggests, it is a question of assuming that the solution function is the sum or the product of functions that depend only on a single variable. In this case, the integration is simplified and the calculation of a solution is feasible. This method makes it possible to find a family of solutions or to calculate a particular solution which depends on boundary conditions and the initial values. The construction of a possible solution allow, on the one hand, to verify a formal calculation and, on the other hand, to test a reasoning numerically before or during a computer programming.

Keywords: PDE, variables, solutions, elliptical, parabolic, hyperbolic

1. Introduction
The one-variable second-order partial differential equations are written in the form

$$A\frac{\partial^2 u}{\partial t^2}+B\frac{\partial^2 u}{\partial t\partial x}+C\frac{\partial^2 u}{\partial x^2}+D\frac{\partial u}{\partial t}+E\frac{\partial u}{\partial x}+F\ u=0$$

Thanks to the theory of characteristics, they are classified into three categories
If B^2-4 A C = 0, the equation is called parabolic.
If B^2-4 A C > 0, the equation is called hyperbolic.
If B^2-4 A C < 0, the equation is called elliptic.

In this article, we will apply the method to particular equations of each category: heat equation, wave equation and Laplace equation.

2. The parabolic PDEs
We will solve a special case of the one-dimensional heat equation. Let's first note the domain represented by $[0, L]$ with L a fixed real. Then, we define the time interval: $t \in [0, +\infty[$. Finally, we define the working domain $Q =]0, T[\times]0, +\infty[$. Let's consider the problem.

$$\left\{\begin{array}{lcll} u\in\mathscr{C}\left(\overline{Q}\right) & & ,\ u\in\mathscr{C}_1^2\left(\overline{Q}\right) & ,\quad (1)\\ \frac{\partial u}{\partial t}-\frac{\partial^2 u}{\partial x^2} & = & 0\ ,\ (t,x)\in Q & ,\quad (2)\end{array}\right. \qquad (1)$$

With $C_1^{\ 2}$ is the space of functions that can be derived twice in space and once in time.

Theorem 1 *The method of separating the variables makes it possible to build a family of possible solutions of class C^∞ on Q of the problem 1.*

Demonstration 2 (Theorem 1) *The idea of the method is to look for a solution of the form*

$$u(x,t) = f(x)g(t) \tag{2}$$

Then the heat equation in 1 is equivalent to

$$f(x)g'(t) = f''(x)g(t) \tag{3}$$

For any (x, t) in Q. Suppose that f and g are not identically zero on Q then,

$$\frac{f''(x)}{f(x)} = \frac{g'(t)}{g(t)}, \quad \forall (x,t) \in Q \tag{4}$$

Therefore, both members of this equation are equal to a certain constant $\lambda \in \mathbb{R}$. We have

$$\begin{cases} f''(x) = \lambda f(x) & , \quad \forall x \in]0,L[\\ g'(t) = \lambda g(t) & , \quad \forall t \in]0,+\infty[\end{cases} \tag{5}$$

Three cases arise

1. If $\lambda > 0$, then $\exists$ A, B $\in \mathbb{R}$ such that

$$\begin{cases} f(x) = A\, e^{\sqrt{\lambda} x} + B\, e^{-\sqrt{\lambda} y} \\ g(t) = g(0)\, e^{\lambda t} \end{cases} \tag{6}$$

2. If $\lambda = 0$ then $\exists$ A, b, c $\in \mathbb{R}$ such that

$$\begin{cases} f(x) = A\,x + b \\ g(t) = c \end{cases} \tag{7}$$

3. If $\lambda < 0$, we put ξ a root of $-\lambda$, then

$$\begin{cases} f(x) = A\cos(\xi x) + B\sin(\xi x) \\ g(t) = g(0)\, e^{-\xi^2 t} \end{cases} \tag{8}$$

Nevertheless, it is not certain that one of these solutions verifies the equality (4) of the system 1.

3. The hyperbolic PDEs

We will solve a special case of the three-dimensional wave equation. Let us first note the spatial domain by the parallelogram R =]0,a[×]0,b[×]0,c[avec a,b,c three real ones fixed. Then, we define the time interval: t ∈ [0,+∞[. Finally, we define the field of work Q = R×]0,+∞[. Let's consider the problem

$$\begin{cases} u \in \mathscr{C}(\overline{Q}) & , \quad u \in \mathscr{C}_1^2(\overline{Q}) \\ \frac{\partial^2 u}{\partial t^2} & = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \quad ,t > 0 \quad , (x,y,z) \in R \end{cases} \tag{9}$$

Theorem 3 The method of separating the variables makes it possible to build a family of possible class solutions C^∞ on Q of the problem 9.

Demonstration 4 (Theorem 3) We are looking for a solution in the form

$$u(x,y,z,t) = X(x)Y(y)Z(z)T(t) \tag{10}$$

The equation will therefore be written

$$XYZT'' = X''YZT + XY''ZT + XYZ''T \tag{11}$$

Dividing the two members by $XYZT \neq 0$ we get

$$\frac{T''}{T} = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} \tag{12}$$

Since the variables are independent, there is a real constant k such that

$$\frac{T''}{T} = k \tag{13}$$

Therefore, $$k = \frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} \tag{14}$$

and for the same reason, there are constants $\alpha, \beta, \gamma \in \mathbb{R}$

$$\begin{cases} \frac{X''}{X} = \alpha \\ \frac{Y''}{Y} = \beta \\ \frac{Z''}{Z} = \gamma \end{cases} \tag{15}$$

Let's solve the equation by X

$$\frac{X''}{X} = \alpha \quad \Rightarrow \quad X'' - \alpha X = 0 \tag{16}$$

Its characteristic equation is r2 = α.

— If $\alpha > 0$, then $\exists$ A, B $\in \mathbb{R}$ such that

$$X(x) = A\, e^{\sqrt{\alpha} x} + B\, e^{-\sqrt{\alpha} x} \tag{17}$$

— If $\alpha = 0$ then $\exists$ A, b $\in$ R such that X(x) = Ax + b.

— If $\alpha < 0$ then $r = \pm i\sqrt{-\alpha}$ where $i^2 = -1$ and $\exists$ $C_1, C_2 \in \mathbb{R}$ such that we have

$$X(x) = C1 \cos(x\sqrt{-\alpha}) + C2 \sin(x\sqrt{-\alpha}) \tag{18}$$

Let's solve the equation by Y

In the same way, we have the family of solutions of the equation in Y:

— If $\beta > 0$, then $\exists$ A, B $\in \mathbb{R}$ such that

$$Y(y) = A\, e^{\sqrt{\beta} y} + B\, e^{-\sqrt{\beta} y} \tag{18}$$

— *If* $\beta = 0$ *then* $\exists A, b \in \mathbb{R}$ *such that* $Y(y) = Ay + b$.
— *If* $\beta < 0$ v $r = \pm i\sqrt{-\beta}$ *where* $i^2 = -1$ *and* $\exists C_1, C_2 \in \mathbb{R}$

such that we have,

$$Y(y) = C_1\, cos\left(y\sqrt{-\beta}\right) + C_2\, sin\left(y\sqrt{-\beta}\right)$$

Let's solve the equation by Z

Also, by a similar reasoning, the family of solutions of the equation in Z is:

— If $\gamma > 0$, then $\exists$ A, B $\in \mathbb{R}$ such that

$$Z(z) = A\, e^{\sqrt{\gamma} z} + B\, e^{-\sqrt{\gamma} z} \tag{19}$$

— *If* $\gamma = 0$ *then* $\exists A, b \in \mathbb{R}$ *such that* $Z(z) = Az + b$.
— *If* $\gamma < 0$ *then* $r = \pm i\sqrt{-\beta}$ *where* $i^2 = -1$ *and* $\exists C_1, C_2 \in \mathbb{R}$

such that we have

$$Z(z) = C_1\, cos\left(z\sqrt{-\beta}\right) + C_2\, sin\left(z\sqrt{-\beta}\right)$$

Let's solve the equation by *T*

The solution of the equation 13, in T, is done in the same way:

$$T'' = kT \tag{20}$$

But taking into consideration the hypothesis

$$k = \alpha + \beta + \gamma \tag{21}$$

Form of the solution *U*

Finally, according to the sign of the constants α, β, γ and k, we can find the general form of a class of possible solutions of the problem 9:

$$u(x,y,z,t) = X(x)Y(y)Z(z)T(t) \tag{22}$$

4, The elliptical PDE

We will solve the one-dimensional Laplace equation. Let us first note the spatial domain by [0,L] with L a real fixed. Then, we define the time interval: $t \in [0, +\infty[$. Finally, we define the domain of work $Q =]0, L[\times]0, +\infty[$. Let's consider the problem

$$\begin{cases} u \in \mathscr{C}\left(\overline{Q}\right) & & u \in \mathscr{C}_1^2\left(\overline{Q}\right) & , & (1) \\ \frac{\partial^2 u}{\partial t^2} + \frac{\partial^2 u}{\partial x^2} & = 0 & , (t,x) \in Q & , & (2) \end{cases} \tag{23}$$

With $C_1 2$ is the space of functions that can be derived twice in space and once in time.

Theorem 5 The method of separating the variables makes it possible to build a family of possible solutions of class C^∞ on Q of the problem 23.

Demonstration 6 (Theorem 5) We will build a family of possible solutions of the PDE 23. For this, we pose

$$u(t,x) = T(t).X(x) \tag{24}$$

This method makes it possible to construct an explicit solution of the Laplace equation. By replacing u(t,x) with its expression in 24, the equality 23 becomes

$$T''(t) + X''(x) = 0 \tag{25}$$

Hence the equality

$$X''(x) = -T''(t) \tag{26}$$

Therefore, there is a constant k such that

$$\begin{cases} X''(x) & = k \\ T''(t) & = -k \end{cases} \tag{27}$$

It is clear that a primitive of the second derivative function X″(x) has a polynomial form

$$X(x) = \frac{k}{2} x^2 + X'(0)x \tag{28}$$

To verify this, it is enough to integrate the equality 27 twice,

$$\begin{aligned} \int_0^x X''(s)\, ds &= \int_0^x k\, ds \\ X'(x) - X'(0) &= kx \end{aligned} \tag{29}$$

Then, we integrate a second time,

$$\begin{aligned} \int_0^x X'(s)\, ds &= \int_0^x (k\, s + X'(0))\, ds \\ &= \left[\frac{k}{2} s^2\right]_0^x + X'(0)\, [s]_0^x \\ &= \frac{k}{2}\, x^2 + X'(0)\, x \end{aligned} \tag{30}$$

By a similar work,

$$T(t) = -\frac{k}{2}\, t^2 + T'(0)\, t \tag{31}$$

By superposition, we have

$$\begin{aligned} u(t,x) &= X(x).T(t) \\ &= \left(\frac{k}{2}\, x^2 + X'(0)\, x\right) \cdot \left(-\frac{k}{2}\, t^2 + T'(0)\, t\right) \end{aligned} \tag{32}$$

This work gives a family of possible solutions explicitly

$$u(t,x) = -\frac{k^2}{4} x^2 t^2 \tag{33}$$

The constant k depends on the boundary conditions of the domain of study. This paragraph allows a simple reconstruction of a solution that can be used, for example, in a numerical verification.

5. Conclusion

The study of partial differential equations or ordinary differential equations does not have a strict method. In this article, we have presented a classical method with application to three examples of equations of different types.

Thus, the method of separation of variables makes it possible to build a family of possible solutions based on the theory of second-order differential equations. Depending on the boundary conditions and the initial temporal conditions, a refinement calculation makes it possible to select the solution of a well posed problem.

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