Modeling of direct and inverse problems

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Article Info

ISSN (online): 2582-7138
Impact Factor: 5.307 (SJIF)
Volume: 05
Issue: 01
January-February 2024
Received: 10-12-2023;
Accepted: 13-01-2024
Page No: 659-662

Abstract
The mathematical study of a phenomenon differs according to the qualitative needs. In this article, the author presents two scientific ways of a mathematical treatment of formal conceptualization. One is part of the study of direct problems, the other of an inverse problems. Examples of models extracted from several bibliographic works are presented.

Keywords: direct problem, inverse problem, hydrogeology, hemodynamics, brain waves, beam theory, Prandtl antenna

1. Introduction

Direct problems have been studied by the scientific community for a long time. In these problems, the causes are known and the effects are the causes are known. So, we are looking for. In this context, it has always been accepted that the same causes produce the same effects. In order to define the notion of inverse problem, M. Kern [1] indicates that according to J.B Keller two problems are said to be inverse of each other if the formulation of one involves the other. However, the roles played by each problems are not symmetrical: whereas in a direct problem we look for effects from the data of the causes, in an inverse problem we look for causes from the data of the effects.

We consider a mathematical problem modeling a physical, medical or economic phenomenon implicitly expressed by

$$G(X,d;p) = 0 \tag{1}$$

With d an answer, X solicitations, p parameters on which the system depends and G representing the operator or the equations of the modeling of the problem considered. In some areas, mathematical modeling is explicit:

$$G(X;p) = d \tag{2}$$

There are two ways of dealing with such a mathematical formulation.

Direct Problems: A direct problem consists in calculating the response d from the data of the solicitations X and the parameters p. This is shown in the diagram 1.

Inverse Problems: In an inverse problem, either we are partially unaware of the system G, or we ignore some parameters p. As compensation, at least partial data are available concerning the output d. The objective is to reconstruct a complete modeling. The word inverse means that we use the model of the equations in reverse: knowing the output, we seek to construct characteristics relating to the stresses or to the parameters of the equations G. This is shown in the diagram 1.

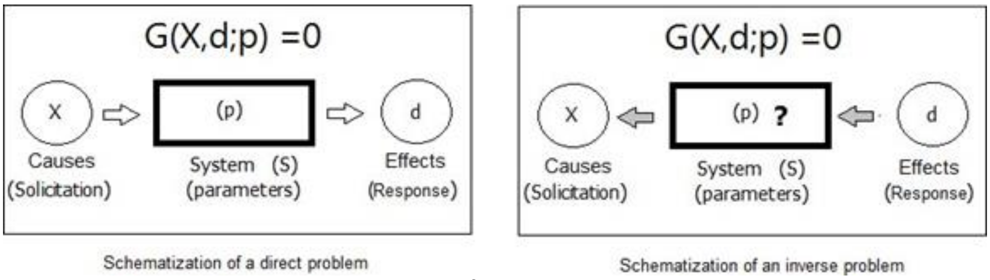

Fig 1: Schematization of direct and inverse problems.

2. Applications

2.1. Application in hydrogeology

This example is taken from the book [2]. Given a solute of concentration C in a groundwater. The solute flows into the water table thanks to the phenomena of diffusion, convection and kinematic dispersion. The evolution of its concentration C over time is modeled by the following equations

$$\left\{\begin{array}{rcll} w\frac{\partial C}{\partial t}+div(-D\nabla C+C\vec{u}) & = & f & \text{in } \Omega_T \\ (-D\nabla C+C\vec{u}).\vec{n} & = & \tau & \text{on } \Sigma_i \\ C & = & 0 & \text{on } \Sigma_m \\ C(x,0) & = & C^0 & \text{in } \Omega \end{array}\right. \quad (3)$$

As an example, we cite the study of the contamination of a groundwater by a radio-active element.

Direct problem: It is a question of studying the evolution of the concentration of the solute over time in the layer of water. This problem is well posed in the sense of Hadamard.

Inverse problem: It is a question of determining the value of the concentration on the boundary of the domain of definition of the water layer. This data completion problem is ill-posed in the sense of Hadamard.

2.2. Application in hemodynamics

In general, the flow of fluids is modeled according to two different approaches. Either we are interested in the movement of a solute in the fluid as in the previous example, or we are interested in the movement of the fluid itself. In this section, we present the model that formalizes the movement of blood. This example is extracted from [3].

The flow of blood through the veins is modeled thanks to the Poiseuille equation. The modeling of the blood flow is done thanks to three variables, *V* the velocity in *m*/*s*, *P* the pressure in N/m^2 and the density ρ in kg/m^3. Each variable describes the fluid at a point of coordinates (x, y, z) in space at a time *t*:

$$\left\{\begin{array}{rcl} V & = & v(x,y,z,t) \\ P & = & p(x,y,z,t) \\ \rho & = & \rho(x,y,z,t) \end{array}\right. \quad (4)$$

In the article [4], the author describes a property of the movement of blood in the arterial duct: there is an axial accumulation. This accumulation generates a symmetry. Hence we can restrict ourselves to a representation in dimensions 2.

As mentioned earlier, blood is a non-Newtonian fluid. In the books [11] and [12], the author presents a modeling of blood, considering it as red blood cells, or particles, which float in plasma. Plasma is a Newtonian fluid. In addition, It is an incompressible fluid. In this work, we consider the Navier-Stokes equation of motion of blood as described above. In the book [5], a description of the transition from the Poiseuille flow equation to the Navier-Stokes equation is detailed in specific cases. The model is described mathematically by the system (5) :

$$\left\{\begin{array}{rcll} \frac{\partial v}{\partial t}+V.div(V) & = & -\frac{1}{\rho}\nabla p\,\mu\,\nabla^2 V & \text{in } \Omega \\ div(V) & = & 0 & \text{on } \Omega\times(0,T) \\ U & = & 0 & \text{on } \partial\Omega\times(0,T) \end{array}\right. \quad (5)$$

With μ the kinematic viscosity of the fluid.

Direct problem: It is a question of studying the speed and the pressure of the blood flow over time. This problem is well posed in the sense of Hadamard as indicated in the article [3].

Inverse problem: We consider an inverse problem of determining a viscosity coefficient in the The Navier-Stokes equation by observation data in a neighborhood of the limit. This problem is studied in detail in the article [7]. This problem is ill-posed in the sense of Hadamard.

2.3. Application on brain waves

This work is extracted from the thesis of [6].

Brain waves are modeled thanks to Maxwell's equations. These are equations that model the electro- magnetic fields (*E*, *B*). The system of Maxwell's equations is as follows:

$$\left\{\begin{array}{rcl} \nabla\times E & = & -\frac{\partial B}{\partial t} \\ \nabla.E & = & \frac{\rho}{\varepsilon_0} \\ \nabla\times B & = & \mu_0 J+\mu_0\varepsilon_0\frac{\partial E}{\partial t} \\ \nabla.B & = & 0 \end{array}\right. \quad (6)$$

with

1. *J* is the current density.
2. ρ is the charge density.
3. ε_0 is the permittivity of the vacuum.

$$\varepsilon_0 = 8.85\,10^{-12}\,Fm^{-1}$$

4. μ_0 is the permeability of the vacuum.

$$\mu_0 = 4\pi\,10^{-7}\,Hm^{-1}$$

Direct Problem: The direct problem consists in studying the evolution of electromagnetic waves over time. This is a well-posed problem in the sense of Hadamard. That is, the direct problem is expressed as: knowing the number, distribution and timing of brain sources, we need to calculate the electromagnetic field. A simplified system of equations is as follows:

$$\begin{cases} \nabla \times E &= 0 \\ \nabla . E &= \frac{\rho}{\varepsilon_0} \\ \nabla \times B &= \mu_0 J \\ \nabla . B &= 0 \end{cases} \quad (7)$$

The scalar potential is denoted by V. In the thesis of G. Adde [6], we find an explicit expression of a solution, in a spherical reference frame:

$$\begin{cases} -\Delta V &= 0 \\ E &= -\nabla V \\ B(r) &= B_0(r) - \frac{\mu_0}{4\pi} \int_\Omega V\left(r'\right) \nabla' \sigma \times \nabla' \left(\frac{1}{\|r-r'\|}\right) dr' \end{cases} \quad (8)$$

Inverse problem: The inverse problem studied in the document [6] is an inverse problem in MEEG within the framework of imaging methods. This is a framework for the reconstruction of a source image at a given instant in such a way that a scalar m represents all the data from the MEG and EEG sensors at that instant and a scalar s designates a bounded variation function defined on S, the surface of the cortex. This is a poorly posed problem in the sense of Hadamard. The problem is reformulated in the form of a minimization problem in a new functional space in order to regularize it.

Application in beam theory

This example is chosen because the mathematical modeling operator does not depend on partial derivatives. It is extracted from [8]. The following figure is considered:

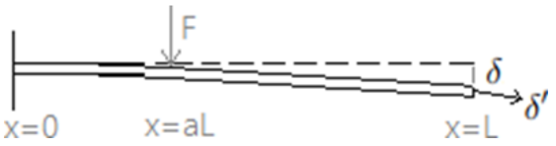

Fig 2: Movement of a beam after a punctual stress

The notations of the figure 2, present a beam of length L, a force denoted F, is exerted at the point $x = aL$. The beam moves by translation δ and rotation δ'.

Direct problem: It is a question of studying the displacement motion of the beam by knowing the force F. This problem is well posed in the sense of Hadamard.

Inverse problem: It is a problem of reconstruction of sources or of solicitation, the position and the intensity of the force (x, F). It is assumed that the translation distance $\delta(L)$ and the angle of rotation $\delta'(L)$ are measured. It is a question of reconstructing the force exerted on the beam. A simple calculation in [8], allows to find:

$$\begin{cases} aL &= 3\left(L - (\delta/\delta')\right) \\ F &= 2EI\delta'(aL)^{-2}. \end{cases} \quad (9)$$

A study of the stability of the force is presented in the document [8]. We give a numerical estimate in the most favorable case.

$$\frac{F_{mesure}}{F_{exact}} - 1 = 0.117 \quad \text{for} \quad a = 0.25 \quad (10)$$

2.4. Application on the Prandtl antenna

This example is chosen from the book [9]. In 1732, the French physicist Henri Pitot proposed a device for measuring the speed of running water and the speed of boats. A Pitot tube is an element of a fluid velocity measurement system. This measuring device was improved by Henry Darcy and then by Ludwig Prandtl. The Prandtlt antenna measures two pressures. One is a static pressure which is the atmospheric pressure in the usual sense of the term, and which depends on the altitude, it is denoted p_s. The other is a total pressure, denoted p_t, which depends on the flow of the fluid and not on atmospheric pressure. This example concerns both the displacement of the air with respect to an object such as an airplane wing or a flow of liquid in a pipe.

Direct problem: It is a question of studying the speed of movement of the fluid by knowing the static pressure and the total pressure. This problem is well posed in the sense of Hadamard. Analytically, the expression for the velocity is:

$$V^2 = \frac{2(p_t - p_s)}{\rho} \quad (11)$$

With V is the fluid velocity, p_t is the total pressure, p_s is the static pressure and ρ is the density of the fluid.

Inverse problem: This is a stress reconstruction problem: the total pressure p_t. In addition, the velocity of the fluid V, the static pressure p_s and the density of the fluid ρ are known.

3. Conclusion

The modeling work does not only concern the mathematical formulation in the form of equations. Indeed, it is a question of properly formally interpreting the causes and effects of an observable phenomenon. Thus, it is necessary to define the objective of the mathematical study: identification, construction of images, and completion of data or illustration of a result. Depending on the target, the problem is qualified as direct or inverse. Bibliographic examples are illustrated.

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