A hybrid nonlinear iterative method for solving nonlinear equations in one and higher dimensions

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Abstract

Scientific computation relies heavily on root finding numerical methods, which allow accurate modelling of complicated systems in physics, engineering, and other fields, resulting in important scientific discoveries. To solve nonlinear equations, this work presents an eighth-order hybrid iterative method that combines a two-step fourth-order strategy with the second-order Newton-Raphson method. With an efficiency rating of 1.5157, this three-step approach evaluates five functions (two functions and three first-order derivatives) and produces results better than some existing iterative methods. Numerical comparisons with proposed method are presented using MAPLE software, and its relevance to real-world models such as computing force between particles and solving Van der Waals equation for volume of a real gas is illustrated. The proposed method is equally suitable for solving both scalar and vector forms of nonlinear equations.

Keywords: Root-finding method; Convergence analysis; Efficiency index; Newton's method

1. Introduction

Iterative methods for computing approximate roots (zeros) of nonlinear equations of the following form:

$$F(u) = 0, \tag{1}$$

are significant in computational and applied mathematics due to their numerous applications in many fields of engineering, mathematical chemistry, biomathematics, physics, and statistics. In the above equation (1), the operator $F : D \subset B \to B'$ is continuous, defined on a nonempty convex subset D of a Banach space B with values in a Banach space B'. In the present research study, we will be looking for an approximation of the local unique solution $\beta \in B$ of the above equation. When it comes to one-dimensional case, then the Banach spaces given in the above equation become $B = B' = \mathbb{R}$. The problem reduces to approximating a simple unique local root β of the following equation:

$$f(u) = 0, \tag{2}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ with I being a neighborhood of β.

Generally, physical, and natural phenomena formulated using models containing nonlinear equations are typically observed in real-world settings including, the medical sciences, physical industries, engineering realm, the triangulation of GPS signals, fluid movement, heat transport, combustion, mathematical epidemiology, and so on. Some of such nonlinear models can be found in (Qureshi, et al., 2023) [10] and (Awadalla, Qureshi, Soomro, & Abuasbeh, 2023) [10].

In the physical and natural sciences, researchers frequently utilize iterative methods to solve nonlinear equations. This is because these equations are often complex to solve accurately, and analytical methods are typically inadequate for addressing these kinds of issues. As a result, an iterative method is necessary. Newton-Raphson (NR) is always seen as the best option because of its simplicity and because it is one of the most well-known and famous iterative algorithms within the scientific community. This algorithm has been widely used for many years to solve non-linear equations. The NR iterative method is also an optimal method in sense of Kung-Traub conjecture (Kung & Traub, 1974) [7]. It offers quadratic convergence with two function evaluations (f and f') per iteration. Mostly, researchers working in the numerical world are trying to adapt a method with a higher order of convergence, but they are also trying to reduce function evaluations per iteration. In this situation, the efficiency index must be discussed, which is defined as $\xi = \rho^{1/\omega}$, where ρ is the order of convergence of the iterative method and ω is the number of function evaluations per iteration.

Due to simplicity, some authors have modified nonlinear iterative methods by carefully merging two different iterative methods with order ρ_1 and ρ_2 and established a hybrid iterative method with an increased order of $\rho_1\rho_2$. Following this approach, we have attempted to develop an iterative method by wisely merging second-order NR and an existing two-step fourth-order method. This type of strategy resulted into a hybrid iterative method having an eight-order of convergence.

2 Existing Hybrid Iterative Methods

In this section, we briefly discuss some well-known iterative methods that are mostly employed to solve both scalar and vector form of nonlinear equations of the type $f(u)=0$. As discussed before, the well-known NR method with two function evaluations per iteration is listed below where we have abbreviated it as NRM:

$$u_{p+1} = u_p - \frac{f(u_p)}{f'(u_p)}, p = 0, 1, 2, \ldots, \tag{3}$$

Where $u_p\,(p=0)$ stands for the initial guess that can be determined via intermediate value theorem for continuous functions. It may also be noted that the initial guess for each method under consideration in this research study is obtained with the said theorem. The efficiency of NRM is about 1.4142.

Authors in (Abro & Shaikh, 2019) [1] proposed a three-step time-efficient hybrid iterative method with sixth-order convergence that requires five function evaluations in one iteration having an efficiency index of about 1.4310 to solve both scalar and vector form of nonlinear equations. The computational steps are shown below:

$$\left.\begin{aligned} v_p &= u_p - \frac{f(u_p)}{f'(u_p)}, \\ w_p &= v_p - \frac{f(v_p)}{f'(v_p)}, \\ u_{p+1} &= v_p - \frac{f(v_p)+f(w_p)}{f'(v_p)}, \end{aligned}\right\} p = 0, 1, 2, \ldots. \tag{4}$$

In (Jaiswal & Choubey, 2013) [5], the authors proposed a new three-step hybrid iterative method (JCM) for solving non-linear equations, with five function evaluations and an efficiency index to be about 1.5157. The iterative method is eighth-order convergent. The computational steps are shown below:

$$\left.\begin{aligned} v_p &= u_p - \frac{f(u_p)}{f'(u_p)}, \\ w_p &= v_p - \frac{2f(u_p)-f(v_p)}{2f(u_p)-5f(v_p)}\left(\frac{f(v_p)}{f'(u_p)}\right), \\ u_{p+1} &= w_p - \frac{f(w_p)}{f'(w_p)}, \end{aligned}\right\} p = 0, 1, 2, \ldots. \tag{5}$$

In (Liu & Wang, 2010) [8], the authors developed an eighth-order convergent hybrid iterative method (LWM) with only four function evaluations required at each iteration and an efficiency index of about 1.6818. It must be noted that the method is optimal in the sense of Kung-Traub conjecture. The conjecture is satisfied as $2^{4-1}=8$. The computational steps are shown below:

$$\left.\begin{aligned} v_p &= u_p - \frac{f(u_p)}{f'(u_p)}, \\ w_p &= v_p - \frac{f(v_p)}{f'(u_p)}\frac{f(u_p)}{f(u_p)-2f(v_p)}, \\ u_{p+1} &= w_p - \frac{f(w_p)}{f'(u_p)}\left[\left\{\frac{f(u_p)-f(v_p)}{f(u_p)-2f(v_p)}\right\}^2 + \frac{f(w_p)}{f(v_p)-5f(w_p)} + \frac{4f(w_p)}{f(u_p)-7f(w_p)}\right], \end{aligned}\right\} p = 0, 1, 2, \cdots \tag{6}$$

In (Neta & Johnson, 2009), the authors presented an iterative method that consists of four-step strategy having eight-order of convergence (NJM) with approximately 1.5157 to be the efficiency index. The computational steps are shown below:

$$\left.\begin{aligned} v_p &= u_p - \frac{f(u_p)}{f'(u_p)}, \\ z_p &= u_p - \frac{f(u_p)}{8f'(u_p)} - \frac{3f(u_p)}{8f'(v_p)}, \\ w_p &= u_p - \frac{6f(u_p)}{f'(u_p)+f'(v_p)+4f'(z_p)}, \\ u_{p+1} &= w_p - \frac{f(w_p)}{f'(u_p)} \frac{f'(u_p)+f'(v_p)-f'(z_p)}{2f'(v_p)-f'(z_p)}, \end{aligned}\right\} p = 0,1,2,\cdots \tag{7}$$

In continuation to the above discussion, several other hybrid iterative methods devised recently can be found in (Jamali, Solangi, & Qureshi, 2022) and (Qureshi, Ramos, & Soomro, 2021).

3. Proposed Hybrid Iterative Method

To decrease the number of function evaluations and increase the order of convergence, several authors have begun with the second-order NR approach and progressed to hybrid-type methods, according to a search of the existing literature. Nevertheless, we made progress by applying the weight function in conjunction with a two-step fourth-order iterative strategy given in (Jaiswal J. , 2014) [4], employing the NR method. The chosen iterative method is as follows:

$$\left.\begin{aligned} v_p &= u_p - \frac{2f(u_p)}{3f'(u_p)}, \\ u_{p+1} &= u_p - \frac{f(u_p)}{2}\left\{\frac{7}{4} - \frac{5f'(v_p)}{4f'(u_p)} + \frac{1}{2}\left(\frac{f'(v_p)}{f'(u_p)}\right)^2\right\}\left\{\frac{1}{f'(u_p)} + \frac{1}{f'(v_p)}\right\}, \end{aligned}\right\} p = 0,1,2,\ldots \tag{8}$$

The above iterative method is now merged with the second-order NR method to yield an eighth-order convergent iterative hybrid type of method. The obtained proposed hybrid iterative method is shown below:

$$\left.\begin{aligned} v_p &= u_p - \frac{f(u_p)}{f'(u_p)}, \\ w_p &= v_p - \frac{2f(v_p)}{3f'(v_p)}, \\ u_{p+1} &= v_p - \frac{f(v_p)}{2}\left\{\frac{7}{4} - \frac{5f'(w_p)}{4f'(v_p)} + \frac{1}{2}\left(\frac{f'(w_p)}{f'(v_p)}\right)^2\right\}\left\{\frac{1}{f'(v_p)} + \frac{1}{f'(w_p)}\right\}, \end{aligned}\right\} p = 0,1,2,\ldots \tag{9}$$

The proposed iterative method presented in (9), hybrid in nature, is a three-step iterative method having five function evaluations per iteration thereby computable efficiency index turns out to be $\xi = \rho^{1/\omega} = 8^{\frac{1}{5}} \approx 1.5157$. The informational efficiency, defined as the ratio of the convergence order and the number of required function evaluations per iteration, of the proposed hybrid iterative method is $\frac{8}{5} = 1.6$.

4. Order of convergence: Taylor's approach

Theorem 1 *Let β be the root of a sufficiently differentiable function $f: I \subset \mathbb{R} \to \mathbb{R}$ on an open interval I. Then, the three-step scheme PNM presented in (9) has an eighth-order convergence, and the error equation is:*

$$e_{p+1} = \frac{f''^4(\beta)\left(79f''^3(\beta) - 18f'(\beta)f''(\beta)f'''(\beta) + f'^2(\beta)f^{(iv)}(\beta)\right)}{3456 f'^7(\beta)} e_p^8 + O(e_p^9), \tag{10}$$

Where $e_p = u_p - \beta$.

Proof

Let β be the simple root of $f(u) = 0$, while u_p be the p^{th} approximation to the root provided by PNM and $e_p = u_p - \beta$ be the error term at the p^{th} iteration.

Using Taylor's expansion for $f(u_p = e_p + \beta)$ about β, we have:

$$f(e_p + \beta) = f'(\beta)e_p + \frac{1}{2!}f''(\beta)e_p^2 + \frac{1}{3!}f'''(\beta)e_p^3 + \frac{1}{4!}f^{(iv)}(\beta)e_p^4 + \frac{1}{5!}f^{(v)}(\beta)e_p^5 + \frac{1}{6!}f^{(vi)}(\beta)e_p^6 + \frac{1}{7!}f^{(vii)}(\beta)e_p^7 + \frac{1}{8!}f^{(viii)}(\beta)e_p^8 + O(e_p^9). \tag{11}$$

Using Taylor's expansion for $f'(u_p = e_p + \beta)$ about β, we have:

$$f'(e_u + \beta) = f'(\beta) + f''(\beta)e_p + \frac{1}{2!}f'''(\beta)e_p^2 + \frac{1}{3!}f^{(iv)}(\beta)e_p^3 + \frac{1}{4!}f^{(v)}(\beta)e_p^4 + \frac{1}{5!}f^{(vi)}(\beta)e_p^5 + \frac{1}{6!}f^{(vii)}(\beta)e_p^6 + \frac{1}{7!}f^{(viii)}(\beta)e_p^7 + \frac{1}{8!}f^{(ix)}(\beta)e_p^8 + O(e_p^9). \quad (12)$$

Multiplying (11) and (12) and putting the result in first step of (9), we have:

$$\sigma_p = \frac{1}{2!}\frac{f''(\beta)}{f'(\beta)}e_p^2 - \frac{1}{3!}\frac{\left(3f''^2(\beta) - 2f'(\beta)f'''(\beta)\right)}{f'^2(\beta)}e_p^3 - \frac{1}{4!}\frac{\left(-12f''^3(\beta) + 14f'(\beta)f''(\beta)f'''(\beta) - 3f'^2(\beta)f^{(iv)}(\beta)\right)}{f'^3(\beta)}e_p^4 + O(e_p^5).$$

where $\sigma_p = v_p - \beta$.

Using Taylor's expansion for $f(v_p = \sigma_p + \beta)$ about β, we have:

$$f(\sigma_p + \beta) = f'(\beta)\sigma_p + \frac{1}{2!}f''(\beta)\sigma_p^2 + \frac{1}{3!}f'''(\beta)\sigma_p^3 + \frac{1}{4!}f^{(iv)}(\beta)\sigma_p^4 + \frac{1}{5!}f^{(v)}(\beta)\sigma_p^5 + \frac{1}{6!}f^{(vi)}(\beta)\sigma_p^6 + \frac{1}{7!}f^{(vii)}(\beta)\sigma_p^7 + \frac{1}{8!}f^{(viii)}(\beta)\sigma_p^8 + O(\sigma_p^9). \quad (13)$$

Using Taylor's expansion for $f'(v_p = \sigma_p + \beta)$ about β, we have:

$$f'(\sigma_p + \beta) = f'(\beta) + f''(\beta)\sigma_p + \frac{1}{2!}f'''(\beta)\sigma_p^2 + \frac{1}{3!}f^{(iv)}(\beta)\sigma_p^3 + \frac{1}{4!}f^{(v)}(\beta)\sigma_p^4 + \frac{1}{5!}f^{(vi)}(\beta)\sigma_p^5 + \frac{1}{6!}f^{(vii)}(\beta)\sigma_p^6 + \frac{1}{7!}f^{(viii)}(\beta)\sigma_p^7 + \frac{1}{8!}f^{(ix)}(\beta)\sigma_p^8 + O(\sigma_p^9). \quad (14)$$

Multiplying (13) and (14) and putting the result in second step of (9), we have:

$$\mathcal{E}_p = \frac{\sigma_p}{3} + \frac{1}{3}\frac{f''(\beta)}{f'(\beta)}\sigma_p^2 - \frac{1}{9}\frac{\left(3f''^2(\beta) - 2f'(\beta)f'''(\beta)\right)}{f'^2(\beta)}e_p^3 - \frac{1}{36}\frac{\left(-12f''^3(\beta) + 14f'(\beta)f''(\beta)f'''(\beta) - 3f'^2(\beta)f^{(iv)}(\beta)\right)}{f'^3(\beta)}e_p^4 + O(e_p^5).$$

Where $\mathcal{E}_p = w_p - \beta$.
Using Taylor's expansion for $f(w_p = \mathcal{E}_p + \beta)$ about β, we have:

$$f(\mathcal{E}_p + \beta) = f'(\beta)\mathcal{E}_p + \frac{1}{2!}f''(\beta)\mathcal{E}_p^2 + \frac{1}{3!}f'''(\beta)\mathcal{E}_p^3 + \frac{1}{4!}f^{(iv)}(\beta)\mathcal{E}_p^4 + \frac{1}{5!}f^{(v)}(\beta)\mathcal{E}_p^5 + \frac{1}{6!}f^{(vi)}(\beta)\mathcal{E}_p^6 + \frac{1}{7!}f^{(vii)}(\beta)\mathcal{E}_p^7 + \frac{1}{8!}f^{(viii)}(\beta)\mathcal{E}_p^8 + O(\mathcal{E}_p^9). \quad (15)$$

Using Taylor's expansion for $f'(w_p = \mathcal{E}_p + \beta)$ about β, we have:

$$f'(\mathcal{E}_p + \beta) = f'(\beta) + f''(\beta)\mathcal{E}_p + \frac{1}{2!}f'''(\beta)\mathcal{E}_p^2 + \frac{1}{3!}f^{(iv)}(\beta)\mathcal{E}_p^3 + \frac{1}{4!}f^{(v)}(\beta)\mathcal{E}_p^4 + \frac{1}{5!}f^{(vi)}(\beta)\mathcal{E}_p^5 + \frac{1}{6!}f^{(vii)}(\beta)\mathcal{E}_p^6 + \frac{1}{7!}f^{(viii)}(\beta)\mathcal{E}_p^7 + \frac{1}{8!}f^{(ix)}(\beta)\mathcal{E}_p^8 + O(\mathcal{E}_p^9). \quad (16)$$

By putting all values in third step of (9), we have:

$$e_{p+1} = \frac{f''^4(\beta)\left(79f''^3(\beta) - 18f'(\beta)f''(\beta)f'''(\beta) + f'^2(\beta)f^{(iv)}(\beta)\right)}{3456f'^7(\beta)}e_p^8 + O(e_p^9). \quad (17)$$

The leading term in the above error equation (17) shows that the proposed three-step hybrid nonlinear iterative method, namely, PMN has an eighth order of convergence. ■

5. Numerical Simulations

5.1. Proposed Method in One Dimension

Numerical simulations of scalar non-linear equations (both academic and physical models) have been discussed in this section. For the comparison purpose, some parameters such as the number of iterations (I), the computational cost ($COC = I \times \omega$), absolute error (AE) at the last iteration, absolute functional value (ϑ) at the last iteration, and CPU time in seconds will remain under consideration. Several iterative methods including the well-know NRM are chosen for the simulations and have been compared with the proposed eighth-order iterative method given in (17). All the numerical computations are carried out in MAPLE 2022 installed in Intel(R) Core (TM) i7 HP laptop having RAM of 24GB and operating at a processing speed of 1.3 GHz. To mention the numerical results, we have set the upper limit for precision at 4,000 digits. The maximum number of iterations allowed to converge towards the required solution is set to be 50. The stopping criterion for the numerical simulations is set as follows:

$$AE = \left|u_{p+1} - u_p\right| \le 10^{-200}.$$

Problem 01: $f_1(x) = x^2 + \sin\left(\frac{x}{5}\right) - \frac{1}{4}$, Required solution = 4.0999e-01.
Problem 02: $f_2(x) = x^3 - x^2 + 3x\cos x - 1$, Required solution = 3.9532e-01.
Problem 03: $f_3(x) = x^5 + x - 10000$, Required solution = 6.3088e+00.

Table 1: Comparison of proposed method with some existing methods for Problem 01 at the initial guess 6.

Method	I	COC	AE	ϑ	Time
NRM	13	26	6.4600e-301	4.1663e-601	1.25e-01
JCM	5	25	9.8043e-486	1.3896e-3886	1.10e-01
LWM	6	24	8.0244e-1597	0	1.09e-01
NJM	10	50	9.4840e-216	3.9910e-431	2.50e-01
PNM	5	25	1.2208e-232	1.2731e-1855	9.40e-02

Table 2: Comparison of proposed method with some existing methods for Problem 02 at the initial guess 0.

Method	I	COC	AE	ϑ	Time
NRM	10	20	1.4608e-346	3.2361e-692	1.09e-01
JCM	4	20	2.2524e-358	1.3994e-2863	9.30e-02
LWM	4	16	1.5607e-330	1.1627e-2638	6.20e-02
NJM	9	45	1.8674e-243	2.3506e-486	1.56e-01
PNM	4	20	1.5519e-312	3.1604e-2495	7.80e-02

Table 3: Comparison of proposed method with some existing methods for Problem 03 at the initial guess 0.

Method	I	COC	AE	ϑ	Time
NRM	44	88	1.3910e-358	4.8586e-713	4.70e-02
JCM	-	-	-	-	-
LWM	-	-	-	-	-
NJM	24	120	9.9150e-392	1.0971e-779	6.30e-02
PNM	16	80	1.2624e-296	4.0085e-2367	1.50e-02

It can be observed in Tables from 1 to 3 that the highest number of iterations in each problem are taken by NRM followed by NJM. The absolute errors computed at the last iteration are smaller in methods of JCM and LWM than the errors by the proposed method but the latter is time inexpensive as shown by the CPU time for Problems 1 and 2. For Problem 3, the methods JCM and LWM diverged as shown in Table 3 while NJM takes more iterations and COC than the proposed methods thereby leading us to conclude that the proposed method is a good choice to be chosen as an iterative method for solving nonlinear equations in one dimension.

The real-life problems are used in several fields (Medical, Science and Engineering). Some of them will be iteratively solved in this section. Physical models namely force acting between particles and Vander Wal equations are chosen from application areas of Physics and Chemistry, respectively. The parameters to compare the chosen iterative methods are same as discussed in the preceding three academic nonlinear problems.

Problem 04: Force acting between particles (Gilat & Subramaniam, 2013)

Consider the following nonlinear model describing the force acting between particles:

$$f(z) = \frac{Qqz}{2\varepsilon_0}\left(1 - \frac{z}{\sqrt{z^2+R^2}}\right) - F, \tag{18}$$

where $\varepsilon_0 = 0.885 \times 10^{-12}\frac{C^2}{Nm^2}$ is the permittivity constant and z is the distance to the particle. The distance z must be determined if $F = 0.3\ N, Q = 9.4 \times 10^{-6}C$, and $q = 2.4 \times 10^{-5}C$, and $R = 0.1m$.

Table 4: Comparison of proposed method with some existing methods for Problem 04 at the initial guess $z_0 = 0.2$.

Method	I	COC	AE	ϑ	Time
NRM	13	26	3.4548e-227	7.8954e-455	4.70e-02
JCM	5	25	1.5201e-602	0	3.20e-02
LWM	5	20	1.2168e-711	3.6400e-3998	7.80e-02
NJM	11	55	4.1571e-296	5.0805e-593	1.10e-01
PNM	6	30	1.1806e-1484	8.6000e-3998	7.80e-02

It can be observed in Table 4 that the proposed hybrid iterative method produced smallest absolute error in comparison to rest of the method for the Problem 4. However, the fewest number of iterations are taken by LWM, but its CPU consumption is equivalent to that of the proposed method. The most expensive method amongst all is NJM with highest COC with reasonably larger CPU time consumption.

Example 05: Volume from Van der Waals equation (Qureshi, et al., 2023)

Johannes Diderik van der Waals came up with the Van der Waals equation in 1873. It adds to the ideal gas law by taking into account the non-negligible dimensions of gas molecules and the forces that exist between them. The model includes adjustment factors for molecule volume and attractive forces, resulting in a more precise representation of the behavior of real gases, particularly under conditions of elevated pressure and reduced temperature. This equation has played a crucial role in enhancing our comprehension of gas characteristics and changes in state. The van der Waals nonlinear condition, a well-known scientific model used in the field of chemical engineering, is stated as follows:

$$\left[P+\frac{n^2k}{V^2}\right](V-nh)=nRT, \tag{19}$$

where k and h, the true positive limits Van der Waals constants which are dependent on the kind of viable gas. The pressure, volume of a real gas, and temperature of the gas are represented by the constants P, V, and T, respectively with n denoting the quantity of moles. The universal gas constant is taken as $R \approx 0.0820578$. The previously stated model can be rewritten as follows:

$$f(V)=PV^3-V^2(RT+hP)n+n^2kV-hkn^3. \tag{20}$$

The above function is a polynomial with third-degree. Some articular values such as k = 16, h = 0.1243, n = 1.29, P = 37atm, and T = 380 ℃ are used for numerical simulations.

Table 5: Comparison of proposed method with some existing methods for Problem 05 at the initial guess $V_0 = 1.1$.

Method	*I*	COC	AE	ϑ	Time
NRM	11	22	7.1720e-217	9.9000e-432	3.10e-02
JCM	4	20	2.9094e-434	3.7252e-3466	1.10e-02
LWM	5	20	2.9771e-454	5.8711e-3627	1.60e-02
NJM	10	50	6.0983e-217	3.1812e-432	1.70e-01
PNM	5	25	6.8358e-537	1.0000e-3999	1.60e-02

It can be observed in Table 5 that the proposed hybrid iterative method produced smallest absolute error in comparison to rest of the methods for the Problem 5. However, the fewest number of iterations are taken by JCM, but its last absolute error is not comparable to that of the proposed method. The proposed method and LWM take same number of iterations and the same CPU time with better absolute error by the preceding one. The most expensive method amongst all is NJM with highest COC with reasonably larger CPU time consumption while the second most expensive, in terms of number of iterations and CPU time, is NRM.

5.2. Proposed Method in Higher Dimensions

The process of finding solutions to systems of nonlinear equations can be intricate, frequently necessitating the use of numerical techniques like the classical NR method. Nonlinear equations, unlike linear systems, consist of functions with curved or complicated forms, rendering straight algebraic solutions impossible. The NR method, an immensely useful iterative technique, proves to be of great value in such situations. It utilizes the notion of linear approximation to progressively improve estimations of the answer through iteration. The method iteratively improves the approximation by starting with an initial estimate and gradually approaching the actual solution via a sequence of tangent lines. Still, the NR method depends a lot on which initial approximation is chosen, and it can run into problems like divergence or convergence to solutions that were not meant to be there. To ensure the reliability and efficiency of this root-finding method in solving systems of nonlinear equations, it is crucial to carefully analyze the mathematical features of the system and take suitable measures. Therefore, higher-order numerical approaches are necessary to estimate the solutions more accurately. In this connection, some authors have developed new strategies for the application of NR method for dealing with the systems of nonlinear equations as can be found in (Ramos & Monteiro, 2017) and (Ramos & Vigo-Aguiar, 2015). The integration of various root-finding approaches to develop hybrid algorithms is currently a subject of ongoing research. Hybrid approaches seek to leverage the advantages of various algorithms to enhance convergence and effectively handle a broader range of mathematical functions. This is what has been done in the present research study.

Problem 06: We consider the following 2-dimensional system of nonlinear equations (Shams, et al., 2021):

$$\boldsymbol{F}(\boldsymbol{X})=\begin{cases} f_1(x_1,x_2)=x_1+e^{x_2}-\cos(x_2) \\ f_2(x_1,x_2)=3x_1-x_2-\sin(x_1) \end{cases} \tag{21}$$

We solve the nonlinear system given in (21) with the proposed hybrid iterative method while taking the initial guess to be $\boldsymbol{X}_0 = (2.5,2.5)$. The exact solution of the system is $\boldsymbol{X} = (0,0)$. For the simulations, the tolerance (stopping criterion) is set to be $\boldsymbol{e}_p = |\boldsymbol{x}_{p+1}-\boldsymbol{x}_p| \le 10^{-200}$, where $p = 1,2,3,\cdots$. The precision digits are pre specified as 4000. The tolerance was achieved at fifth iteration at which the normed absolute error value turns out to be 2.1107e-461 with function's value very close to 0 as shown by the last column of Table 6. The CPU time (measures in seconds) consumed by the proposed hybrid iterative method to obtain the numerical simulations for the nonlinear system given in Problem 06 is 1.5000e-02s.

Table 6: Numerical solutions with proposed hybrid iterative method for Problem 06 at the initial guess $\boldsymbol{X}_0 = (2.5,2.5)$.

Iteration	$\boldsymbol{e}_p = \lvert \boldsymbol{x}_{p+1} - \boldsymbol{x}_p \rvert$	Approximation to $\boldsymbol{F}(\boldsymbol{X})$
1	2.4021e+00	$(9.7851e-02, 1.7975e-01)$
2	1.7975e-01	$(4.0378e-08, 8.2189e-08)$
3	8.2189e-08	$(1.6115e-58, 3.2801e-58)$
4	3.2801e-58	$(1.0370e-461, 2.1107e-461)$
5	2.1107e-461	$(3.0484e-3687, 6.2048e-3687)$

In the light of numerical simulations, it can be said that the hybrid iterative method (9) with convergence order eight can be employed to solve both univariate and multivariate forms of nonlinear equations.

6. Conclusion

To summarize, this study focuses on the crucial significance of root-finding numerical techniques in scientific computation. It specifically presents a novel eighth-order hybrid iterative method for effectively solving nonlinear equations. The approach utilizes a two-step fourth-order technique in conjunction with the second-order Newton-Raphson method, resulting in an efficiency rating of 1.5157. Comparisons with existing methodologies employing MAPLE show its higher performance through the examination of five nonlinear equations. The method's practical significance is demonstrated through its application in real-world models, such as calculating the force between particles and solving the Van der Waals equation to determine the volume of a real gas. The results highlight the effectiveness of the strategy in enhancing scientific computation in several fields.

Declarations

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Ethical Approval: Not applicable.

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References

1. Abro HA, Shaikh MM. A new time-efficient and convergent nonlinear solver. Appl Math Comput. 2019; 355:516-536. doi:10.1016/j.amc.2019.03.012
2. Awadalla M, Qureshi S, Soomro A, Abuasbeh K. A Novel Three-Step Numerical Solver for Physical Models under Fractal Behavior. Symmetry, 2023, 15(2). doi:10.3390/sym15020330
3. Gilat A, Subramaniam V. Numerical Methods for Engineers and Scientists: An Introduction with Applications Using MATLAB. 3rd ed. Wiley Publishing; 2013.
4. Jaiswal J. Some Class of Third- and Fourth-Order Iterative Methods for Solving Nonlinear Equations. J Appl Math. 2014, 2014. doi:10.1155/2014/817656
5. Jaiswal JP, Choubey N. A New Efficient Optimal Eighth-Order Iterative Method for Solving Nonlinear Equations. math.NA. doi:https://arxiv.org/abs/1304.4702
6. Jamali K, Solangi M, Qureshi S. A novel hybrid iterative method for applied mathematical models with time-efficiency. J Appl Math Comput Mech. 2022; 21(3):19-29. doi:10.17512/jamcm.2022.3.02
7. Kung HT, Traub JF. Optimal order of one-point and multipoint iteration. J Assoc Comput Mach. 1974; 21(4):643-651. doi:10.1145/321850.321860
8. Liu L, Wang X. Eighth-order methods with high efficiency index for solving nonlinear equations. Appl Math Comput. 2010; 215(9):3449-3454. doi:10.1016/j.amc.2009.10.040
9. Neta B, Johnson A. High order nonlinear solver. J Comput Methods Sci Eng. 2009; 8(4-6):245-250. doi:10.3233/JCM-2008-84-602
10. Qureshi S, Argyros IK, Soomro A, Gdawiec K, Shaikh AA, Hincal E. A new optimal root-finding iterative algorithm: local and semilocal analysis with polynomiography. Numer Algorithms. 2023:1-31. doi:10.1007/s11075-023-01625-7
11. Qureshi S, Ramos H, Soomro A. A New Nonlinear Ninth-Order Root-Finding Method with Error Analysis and Basins of Attraction. Mathematics. 2021; 9(16). doi:10.3390/math9161996
12. Ramos H, Monteiro M. A new approach based on the Newton's method to solve systems of nonlinear equations. J Comput Appl Math. 2017; 318:3-13. doi:10.1016/j.cam.2016.12.019
13. Ramos H, Vigo-Aguiar J. The application of Newton's method in vector form for solving nonlinear scalar equations where the classical Newton method fails. J Comput Appl Math. 2015; 275:228-237. doi:10.1016/j.cam.2014.07.028
14. Shams M, Rafiq N, Kausar N, Agarwal P, Park C, Mir N. On iterative techniques for estimating all roots of nonlinear equation and its system with application in differential equation. Adv Differ Equ., 2021, 1-18. doi:10.1186/s13662-021-03636-x