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Mathematical analysis of a unit tetra-shaped finite element mesh in discretized solid

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Abstract

The stability and deformation of a solid depends on the variability in its primary field variables, referred to as displacement field. Displacement, being the change in the position of particles at the atomic and sub-atomic levels within a solid; when, known, enables easier determination of the rate of deformation, and the force intensity within a solid. And the mathematical analysis of this cumbersome computational process has always being presented as a literature with little or no analytic details. Yes finite element analysis is a numeral method of solving a differential equation, a process where a solid of any shape is discretized into small coordinate locations in space called mesh, which contains nodes that represent the shape of the whole geometry. A structure may contain thirty thousand mesh, but its displacement field can only be determined by the computational analysis of a single mesh. The rest will be the assembling of the whole into a system. All these descriptions had always being in words. Here, we give a detailed mathematical analysis to this process by considering a unit tetra-shaped finite element mesh.

Keywords: Solid, Mesh, Tetra, Finite Element, Displacement, Elasticity, Functional, Continuum

1. Introduction

Finite element method is a method of finding approximate solution to a wide range of problems spanning from engineering, biomechanical, stress distribution, and heat conduction capabilities of a solid. The analysis is usually achieved by discretizing the structural material into smaller geometric unit called mesh, which enables the determination of the field variable such as displacement, stress, strain and the general material response to load at unit bases. The generic applicability of finite element method to science and engineering makes it useful to scholars and researchers, who depend on it for solving complex partial differential equations generated from structural analyses.

Notable publication in this area are [1], finite element analysis and light weight optimization design on main frame structure of large electrostatic precipitator [2]. Worked, on finite element analysis of two different dental implants; stress distribution in the prosthesis, abutment, implant, and supporting bones. Discussion on the evaluation of parameters of an osseo-integrated dental implants using finite element analysis, a two-dimensional comparative study, examining the effects of implant was carried out in [3]. While [4, 11] addressed, finite element stress analysis of dental prostheses supported by straight and angle implant. The influence of restoration type on stress distribution in bone around implants a three-dimensional finite analysis was solved in [5]. Also, three-dimensional finite element analysis of biomechanical behaviors of implants with different connections, lengths, and diameters placed in the maxillary was discussed in [6, 12, 13].

Even in civil and structural engineering, finite element analysis is used to determine the structural integrity before the foundations are laid [7, 8, 9, 10]. In all these applications of finite element analysis in different fields of studies, its fundamentals, still lies in the mathematical analysis of the unit element, whose details had not been reported in literature. Here, in this paper we give an explicit mathematical analysis of the finite element problem solving process by considering ‘’a tetra-shaped mesh’’ of a discretized solid under concentrated load.

2. Problem Formulation

A cross-sectional area A, of a concrete wall is discretized into nth-thousand mesh of $l\ squared$ centimeter each. Suppose the wall is subjected to concentrated load q , and is under the influence of body forces. As shown in Figure 1. Here, we wish to determine the mathematical computational principle of finite element analysis, in determining the primary field variable (displacement) of the wall.

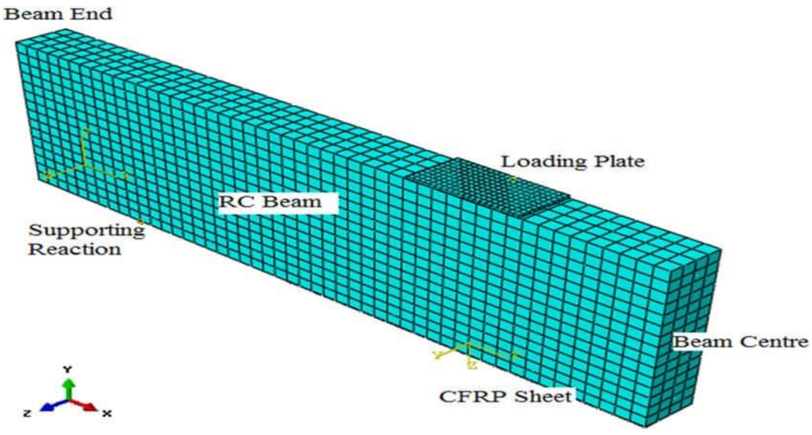

Fig 1: A discretized concrete wall

3. Mathematical Formulations

The deformation, and stationarity of a continuum is determined by the variation in the displacement field of its functional. The functional in the case of a solid body will be derived from its elastic property [14]. Mathematically defined as:

$$Elasticity(E) = \frac{Stress}{Strain} \tag{1}$$

where

$$Stress = \frac{Force}{Area} = \frac{F}{A} \tag{2}$$

and

$$Strain = \frac{Extension}{Length} = \frac{e}{l} \tag{3}$$

From equation (2) and (3), we can rewrite equation (1) analytically as:

$$E = \frac{Fl}{Ae} \tag{4}$$

$$Fl = AEe = AE\frac{du}{dx} \ (5)$$

Where u is the displacement caused by the extension along the $x, and\ y - axis$ for a two dimensional body. Hence, the strain energy stored in a solid, is describe as a kinetic energy, due to the continuous displacement of it atomic and sub-atomic particles. This takes the form analytically [15] as

$$K.e\ = {}^1\!/_2\, AE\left(\frac{du}{dx}\right)^2 \tag{6}$$

and

$$K.e = {}^1\!/_2\, AE\left(\frac{du}{dy}\right)^2 \tag{7}$$

respectively for the $x, and\ y -$directions for two dimensional bodies.

If the body is subjected to external load q; then, the functional(F) of the continuum is equivalent to the strain energy plus the external load times displacement. That is,

$$F = {}^1\!/_2\, AE\left(\frac{du}{dx}\right)^2 - qu \tag{8}$$

And

$$F = {}^1\!/_2\, AE\left(\frac{du}{dy}\right)^2 - qu \tag{9}$$

III Condition for Extrema

The necessary condition for a solid to deform or remain stationary, is equivalent to finding a differential equation that represents the functional (F). This condition is achieved by using calculus of variation to determine the displacement field of the functional. Calculus of variation deals primarily with finding maximum or minimum value of a definite integral involving a functional [16, 17]. A simple example of a functional is the shortest distance of a curve through two points $a(x_1, u_a)$ and $b(x_2, u_b)$, which a displaced particle can travel within a solid shown in Figure 2 and mathematically represented by (10).

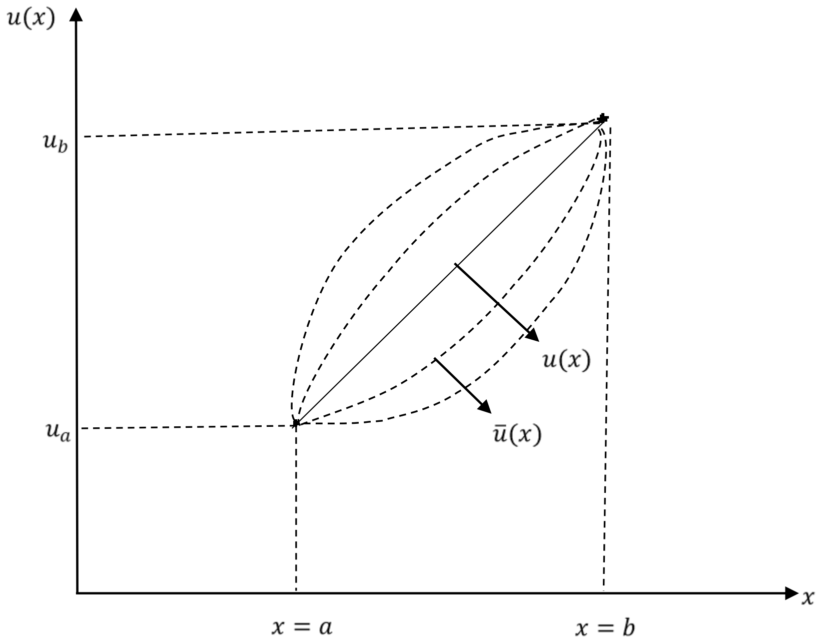

Fig 2: Displacement Part-lines Travelled by Particle within a Solid

$$I[u(x)] = \int_a^b F\left(u, \frac{du}{dx}, x\right) dx \tag{10}$$

$$\bar{u}(x) = u(x) + \varepsilon \tag{11}$$

where ε is a small parameter, and the difference between $\bar{u}(x)$ and $u(x)$ is called the variation in the displacement field $u(x)$; denoted by,

$$\delta u(x) = \bar{u}(x) - u(x) \tag{12}$$

The difference between du and δu at a given point x:
The variation δu refers to the difference between $\bar{u}(x)$ and $u(x)$; while, du refers to the incremental change in $u(x)$ as x changes to $x + dx$
and we define

$$\delta(u') = \bar{u}'(x) - u'(x) = [\delta u]' \tag{13}$$

where $(\ )'$ denotes differentiation with respect to x.
For a given x, as we move from $u(x)$ to $\bar{u}(x)$; following figure 2,

$$\Delta F = F\left(\bar{u}, \overline{u}', x\right) - F(u, u'x) = F(u + \delta u, u' + \delta u', x) - F(u, u'x) \tag{14}$$

expanding the first term in equation (14) by Taylor series [18]; we get,

$$F(u + \delta u, u' + \delta u', x) = F(u, u'x) + \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'\right) + \frac{1}{2!}\left(\frac{\partial^2 F}{\partial u^2}\delta u^2 + 2\frac{\partial^2 F}{\partial u \partial u\prime}\delta u \delta u' + \frac{\partial^2 F}{\partial u\prime^2}\delta u'^2\right) \tag{15}$$

hence

$$\Delta F = \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'\right) + \frac{1}{2!}\left(\frac{\partial^2 F}{\partial u^2}\delta u^2 + 2\frac{\partial^2 F}{\partial u \partial u\prime}\delta u \delta u' + \frac{\partial^2 F}{\partial u\prime^2}\delta u'^2\right) \tag{16}$$

The first variation of F is defined as:

$\delta F = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'$ (17)

and the second variation of F is:

$$\delta^2 F = \frac{\partial^2 F}{\partial u^2}\delta u^2 + 2\frac{\partial^2 F}{\partial u \partial u'}\delta u \delta u' + \frac{\partial^2 F}{\partial u'^2}\delta u'^2 \quad (18)$$

So that,

$$\Delta F = \delta F + \frac{1}{2!}\delta^2 F + \cdots + \frac{1}{n!}\delta^n F \quad (19)$$

Studying what happens to the variation I, in the neighbourhood of $u(x)$; that is,

$$\Delta I = I(\bar{u}, \bar{u}', x) - I(u, u'x) = \int_a^b F(\bar{u}, \bar{u}', x)dx - \int_a^b F(u, u'x)dx \quad (20)$$

$$\int_a^b \Delta F \, dx = \int_a^b \left(\delta F + \frac{1}{2!}\delta^2 F + \cdots\right) dx \quad (21)$$

Then, the first variation of the functional within the solid is defined by:

$$\delta I = \int_a^b \delta F \, dx \quad (22)$$

and the second variation is given as:

$$\delta^2 I = \int_a^b \delta^2 \, F dx \quad (23)$$

So that

$$\Delta I = \delta I + \frac{1}{2!}\delta^2 I \quad (24)$$

If I represents the total potential of the structure, and we are searching for a stable equilibrium configuration; then, we wish to find $u(x)$ that minimizes the variation I. Since $u(x)$, minimizes I; then, $\Delta I \to 0, when\ \bar{u}(x) = u(x)$. Then I attains a minimum [17].

From equation (17) and (22); we have,

$$\delta I = \int_a^b \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'\right) dx \quad (25)$$

Recall from equation (13); that is,

$$\delta(u') = (\delta u') = \frac{d}{dx}\delta u \quad (26)$$

Substituting for the second term in equation (25) with equation (26); we have,

$$\delta I = \int_a^b \left(\frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\frac{d}{dx}\delta u\right) dx \quad (27)$$

We then isolate the second term in equation (27) for further analysis.

$$\int_a^b \frac{\partial F}{\partial u'}\frac{d}{dx}\delta u dx = \int_a^b \frac{\partial F}{\partial u'} d(\delta u) \quad (28)$$

Performing integration by part on the second term of equation (28); we get,

$$\int_a^b \frac{\partial F}{\partial u'} d(\delta u) = \left[\frac{\partial F}{\partial u'}\delta u\right]_a^b - \int_a^b (\delta u)\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) dx \quad (29)$$

Considering the boundary values $\delta u(a) = \delta u(b) = 0$, since the end-points are usually fixed. Leaving equation (29) as,

$$\int_a^b \frac{\partial F}{\partial u'} d(\delta u) = -\int_a^b (\delta u)\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) dx \quad (30)$$

Hence, replacing the second term of equation (27) with the second term of equation (30); we have,

$$\delta I = \int_a^b \left(\frac{\partial F}{\partial u}\delta u dx - (\delta u)\frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) dx\right) \quad (31)$$

Factoring out the common terms from equation (31); gives,

$$\delta I = \int_a^b \left(\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right)\right)\delta u dx \tag{32}$$

$u(x)$ minimizes I when $\bar{u}(x) = u(x);$ implying, $\delta I = 0$. Making equation (32)

$$\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = 0 \tag{33}$$

Equation (33) is the famous Euler-Lagrange equation of solid mechanics [15]. Which is the necessary condition for a continuum to be stationary.
The principle of stationary total potential (PSTP), describing the equilibrium configuration of a solid states, that the problem of finding $u(x)$, that makes I stationary with respect to small variation in $u(x)$, is equivalent to the problem of finding $u(x)$ that satisfies the governing differential equation for the problem [15] Hence we use equation (33) to convert the functional equation (equation (8)) into a governing differential equation as follows:
Differentiating the functional equation $F = \frac{1}{2}AE\left(\frac{du'}{dx}\right)^2 - qu$, with respect to u; that is,

$$\frac{\partial F}{\partial u} = -q \tag{34}$$

also, differentiating the functional equation $F = \frac{1}{2}AE\left(\frac{du}{dx}\right)^2 - qu$, with respect to u'; we have,

$$\frac{\partial F}{\partial u'} = AE\frac{du}{dx} \tag{35}$$

Substituting equation (34) and (35) into $\frac{\partial F}{\partial u} - \frac{d}{dx}\left(\frac{\partial F}{\partial u'}\right) = 0$, to obtain the differential equation of the functional equivalence as:

$$-q - \frac{d}{dx}\left(AE\frac{du}{dx}\right) = 0$$

resulting to,

$$AE\frac{d^2u}{dx^2} + q = 0 \tag{36}$$

and by a similar procedure; we have,

$$AE\frac{d^2u}{dy^2} + q = 0 \tag{37}$$

Equation (36) and (37) is a 1-dimensional governing differential equation representing the functional in x and y directions while the 2-dimensional case is obtained by summing equation (36) and (37). That is,

$$AE\frac{d^2u}{dx^2} + q + AE\frac{d^2u}{dy^2} - q = 0$$

Leading to a Laplace equation of the form;

$$AE\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 \tag{38}$$

IV. Finite Element Method

For computational purpose, let the displacement $u(x, y)$ be denoted by $\psi(x, y)$.
So that equation (38) can be rewritten as;

$$AE\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) = 0 \tag{39}$$

and the weak form of equation (39) becomes

$$AE\iint W\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2}\right) dA = 0 \tag{40}$$

by integration by parts, the first term in the bracket becomes,

$$\int_{x_l}^{x_u} W\frac{d^2\psi}{dx^2}dx = \left(W\frac{d\psi}{dx}\right)_{x_l}^{x_u} - \int_{x_l}^{x_u}\frac{dW}{dx}\frac{d\psi}{dx}dx \tag{41}$$

On a similar lines, we write (using the Green Gauss Theorem)

$$\iint W\frac{\partial^2\psi}{\partial x^2}dA = \int_{y_l}^{y_u}\left(W\frac{\partial\psi}{\partial x}\right)_{x_l}^{x_u}dy - \iint\frac{\partial W}{\partial x}\frac{\partial\psi}{\partial x}dA \tag{42}$$

The general interpretation of $x_l, x_u, and\ y_l, y_u$; [15] .Therefore,

$$dy = \pm l_x ds \tag{43}$$

where l_x is the direction cosine of the out-ward normal $\vec{n}$, and $\pm$ is introduced to take care of both ends.

$$\int_{y_l}^{y_u}\left(W\frac{\partial\psi}{\partial x}\right)_{x_l}^{x_u}dy = \oint W\frac{\partial\psi}{\partial x}l_x ds \tag{44}$$

Replacing the second term of equation (42) with equation (44); we get,

$$\iint W\frac{\partial^2\psi}{\partial x^2}dA = -\iint\frac{\partial W}{\partial x}\frac{\partial\psi}{\partial x}dA + \oint W\frac{\partial\psi}{\partial x}l_x ds \tag{45}$$

and by a similar procedure; we have,

$$\iint W\frac{\partial^2\psi}{\partial y^2}dA = -\iint\frac{\partial W}{\partial y}\frac{\partial\psi}{\partial y}dA + \oint W\frac{\partial\psi}{\partial y}l_y ds \tag{46}$$

hence

$$AE\iint W\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}\right)dA = AE\left[-\iint\left(\frac{\partial W}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial W}{\partial y}\frac{\partial\psi}{\partial y}\right)dA + \oint W\frac{\partial\psi}{\partial x}l_x ds + \oint W\frac{\partial\psi}{\partial y}l_y ds\right] = 0$$

Therefore

$$AE\iint\left(\frac{\partial W}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial W}{\partial y}\frac{\partial\psi}{\partial y}\right)dA = AE\oint W\left(\frac{\partial\psi}{\partial x}l_x+\frac{\partial\psi}{\partial y}l_y\right)ds = AE\oint W\left(\frac{\partial\psi}{\partial n}\right)ds$$

$$AE\iint\left(\frac{\partial W}{\partial x}\frac{\partial\psi}{\partial x}+\frac{\partial W}{\partial y}\frac{\partial\psi}{\partial y}\right)dA = AE\oint W\left(\frac{\partial\psi}{\partial n}\right)ds \tag{47}$$

$$\left(AE\iint[B]^T[B]dA\right)\{\psi\} = AE\oint[N]^T\left(\frac{\partial\psi}{\partial n}\right)ds \tag{48}$$

$$[k]\{\psi\} = \{f\} \tag{49}$$

Where the stiffness matrix,

$$[k] = AE\iint[B]^T[B]dA$$

and the loading force,

$$\{f\} = AE\oint[N]^T\left(\frac{\partial\psi}{\partial n}\right)ds$$

Then, for the entire finite element mesh; we have,

$$AE\sum_1^{NELEM}\left(\iint[B]^T[B]dA\right)\{\psi\} = AE\sum_1^{NELEM}\oint[N]^T\left(\frac{\partial\psi}{\partial n}\right)ds \tag{50}$$

where [N] and $[N]^T$ are the shape function matrix and its transpose respectively. [B] and $[B]^T$ are the derivatives of the shape function while $\{\psi\}$ is the displacement vector.
Note: In the Galerkin formulation, we use the same shape function as the weight function $(W = N)$ [15].

V. Computation of Shape Function

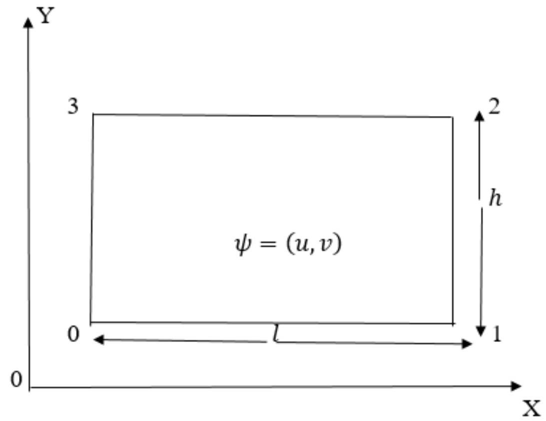

Fig 3: Displacement Distribution in Rectangular Mesh.

Let the axial and transverse displacement in the quadrilateral be represented by u *and* v respectively.
Here, the unknown field variables vary independently along two directions. Hence, we assume that the displacement field over the mesh is given by:

$$u(x, y) = C_0 + C_1x + C_2y + C_3xy \tag{51}$$

and

$$v(x, y) = C_0 + C_1x + C_2y + C_3xy \tag{52}$$

Thus, for the rectangular element of size $(l \times h)$. On computation; we have,

$$u_1 = C_0 \tag{53}$$

$$u_2 = C_0 + C_1l \tag{54}$$

$$u_3 = C_0 + C_1l + C_2h + C_3lh \tag{55}$$

$$u_4 = C_0 + C_4h \tag{56}$$

Solving for, $C_o, C_1, C_2, and\ C_3$; we have,

$$C_0 = u_1 \tag{57}$$

$$C_1 = \frac{u_2 - u_1}{l} \tag{58}$$

$$C_2 = \frac{u_4 - u_1}{h} \tag{59}$$

$$C_3 = \frac{u_3 + u_1 - u_2 - u_4}{lh} \tag{60}$$

Substituting for, $C_o, C_1, C_2, and\ C_3$; in equation (51); we have,

$$u(x, y) = u_1 + \left(\frac{u_2 - u_1}{l}\right)x + \left(\frac{u_4 - u_1}{h}\right)y + \left(\frac{u_3 + u_1 - u_2 - u_4}{lh}\right)xy \tag{61}$$

by expanding and grouping like terms; we have,

$$u(x, y) = \left(1 - \frac{x}{l} - \frac{y}{h} + \frac{xy}{lh}\right)u_1 + \left(\frac{x}{l} - \frac{xy}{lh}\right)u_2 + \left(\frac{xy}{lh}\right)u_3 + \left(\frac{y}{h} - \frac{xy}{lh}\right)u_4 \tag{62}$$

In standard finite element notation equation (62) is written as:

$$u(x, y) = [N_1, N_2, N_3, N_4]\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} \tag{63}$$

By a similar procedure; we have

$$v(x,y) = [N_1, N_2, N_3, N_4]\begin{Bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{Bmatrix} \tag{64}$$

where

$$N_1 = 1 - \frac{x}{l} - \frac{y}{h} + \frac{xy}{lh}$$

$$N_2 = \frac{x}{l} - \frac{xy}{lh}$$

$$N_3 = \frac{xy}{lh}$$

$$N_4 = \frac{y}{h} - \frac{xy}{lh}$$

Hence, N_i is the shape function, describing the uniform distribution of the displacement field. Since Figure 1; is used to model structural mechanics problems, each node will have two degrees of freedom viz: $u \; and \; v$; hence, we can write the displacement field, using the shape functions derived in equation (63) and (64) as:

$$[N_i]\{\psi\} = [N_i]\begin{Bmatrix} u_i \\ v_i \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} \tag{65}$$

and

$$\begin{bmatrix} \frac{dN_i}{dx} \\ \frac{dN_i}{dy} \end{bmatrix} = [B] = \begin{bmatrix} \frac{dN_1}{dx} & 0 & \frac{dN_2}{dx} & 0 & \frac{dN_3}{dx} & 0 & \frac{dN_4}{dx} & 0 \\ 0 & \frac{dN_1}{dy} & 0 & \frac{dN_2}{dy} & 0 & \frac{dN_3}{dy} & 0 & \frac{dN_4}{dy} \end{bmatrix}$$

then;

$$[B]^T[B] = \begin{bmatrix} \frac{dN_1}{dx} & 0 \\ 0 & \frac{dN_1}{dy} \\ \frac{dN_2}{dx} & 0 \\ 0 & \frac{dN_2}{dy} \\ \frac{dN_3}{dx} & 0 \\ 0 & \frac{dN_3}{dy} \\ \frac{dN_4}{dx} & 0 \\ 0 & \frac{dN_4}{dy} \end{bmatrix}\begin{bmatrix} \frac{dN_1}{dx} & 0 & \frac{dN_2}{dx} & 0 & \frac{dN_3}{dx} & 0 & \frac{dN_4}{dx} & 0 \\ 0 & \frac{dN_1}{dy} & 0 & \frac{dN_2}{dy} & 0 & \frac{dN_3}{dy} & 0 & \frac{dN_4}{dy} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{dN_1}{dx}\frac{dN_1}{dx} & 0 & \frac{dN_1}{dx}\frac{dN_2}{dx} & 0 & \frac{dN_1}{dx}\frac{dN_3}{dx} & 0 & \frac{dN_1}{dx}\frac{dN_4}{dx} & 0 \\ 0 & \frac{dN_1}{dy}\frac{dN_1}{dy} & 0 & \frac{dN_1}{dy}\frac{dN_2}{dy} & 0 & \frac{dN_1}{dy}\frac{dN_3}{dy} & 0 & \frac{dN_1}{dy}\frac{dN_4}{dy} \\ \frac{dN_2}{dx}\frac{dN_1}{dx} & 0 & \frac{dN_2}{dx}\frac{dN_2}{dx} & 0 & \frac{dN_2}{dx}\frac{dN_3}{dx} & 0 & \frac{dN_2}{dx}\frac{dN_4}{dx} & 0 \\ 0 & \frac{dN_2}{dy}\frac{dN_1}{dy} & 0 & \frac{dN_2}{dy}\frac{dN_2}{dy} & 0 & \frac{dN_2}{dy}\frac{dN_3}{dy} & 0 & \frac{dN_2}{dy}\frac{dN_4}{dy} \\ \frac{dN_3}{dx}\frac{dN_1}{dx} & 0 & \frac{dN_3}{dx}\frac{dN_2}{dx} & 0 & \frac{dN_3}{dx}\frac{dN_3}{dx} & 0 & \frac{dN_3}{dx}\frac{dN_4}{dx} & 0 \\ 0 & \frac{dN_3}{dy}\frac{dN_1}{dy} & 0 & \frac{dN_3}{dy}\frac{dN_2}{dy} & 0 & \frac{dN_3}{dy}\frac{dN_3}{dy} & 0 & \frac{dN_3}{dy}\frac{dN_4}{dy} \\ \frac{dN_4}{dx}\frac{dN_1}{dx} & 0 & \frac{dN_4}{dx}\frac{dN_2}{dx} & 0 & \frac{dN_4}{dx}\frac{dN_3}{dx} & 0 & \frac{dN_4}{dx}\frac{dN_4}{dx} & 0 \\ 0 & \frac{dN_4}{dy}\frac{dN_1}{dy} & 0 & \frac{dN_4}{dy}\frac{dN_2}{dy} & 0 & \frac{dN_4}{dy}\frac{dN_3}{dy} & 0 & \frac{dN_4}{dy}\frac{dN_4}{dy} \end{bmatrix} \quad (66)$$

Where

$$\frac{dN_1}{dx} = \frac{y}{l^2} - \frac{1}{l}, \frac{dN_2}{dx} = \frac{1}{l} - \frac{y}{l^2}, \frac{dN_3}{dx} = \frac{y}{l^2}, \frac{dN_4}{dx} = \frac{y}{l^2}$$

And

$$\frac{dN_1}{dy} = \frac{x}{l^2} - \frac{1}{l}, \frac{dN_2}{dy} = -\frac{x}{l}, \frac{dN_3}{dy} = \frac{x}{l^2}, \frac{dN_4}{dy} = \frac{1}{l} - \frac{x}{l^2}$$

Then,

$$\frac{dN_1}{dx}.\frac{dN_1}{dx} = \left(\frac{y}{l^2} - \frac{1}{l}\right)\left(\frac{y}{l^2} - \frac{1}{l}\right) = \frac{y^2}{l^4} - \frac{2y}{l^3} + \frac{1}{l^2}$$

Hence, for $\int_0^l \int_0^1 [B]^T [B] dxdy$, compute for each component in matrix in equation (68) as follows:

$$\int_0^l \int_0^1 \frac{dN_1}{dx}.\frac{dN_1}{dx} dxdy = \int_0^l \int_0^1 \left(\frac{y^2}{l^4} - \frac{2y}{l^3} + \frac{1}{l^2}\right) dxdy = 0.3l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_2}{dx}.\frac{dN_1}{dx} dxdy = \int_0^l \int_0^1 \left(\frac{2y}{l^3} - \frac{1}{l^2} + \frac{y^2}{l^4}\right) dxdy = 0.3l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_3}{dx}.\frac{dN_1}{dx} dxdy = \int_0^l \int_0^1 \left(\frac{y^2}{l^4} - \frac{y}{l^3}\right) dxdy = -0.2l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_4}{dx}.\frac{dN_1}{dx} dxdy = \int_0^l \int_0^1 \left(\frac{y^2}{l^4} - \frac{y}{l^3}\right) dxdy = -0.2l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_1}{dy}.\frac{dN_1}{dy} dydx = \int_0^l \int_0^1 \left(\frac{x^2}{l^4} - \frac{2x}{l^3} + \frac{1}{l^2}\right) dydx = 0.3l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_2}{dy}.\frac{dN_1}{dy} dydx = \int_0^l \int_0^1 \left(-\frac{x^2}{l^3} + \frac{x}{l^2}\right) dydx = 0.2$$

$$\int_0^l \int_0^1 \frac{dN_3}{dy}.\frac{dN_1}{dy} dydx = \int_0^l \int_0^1 \left(\frac{x^2}{l^4} - \frac{x}{l^3}\right) dydx = 0.2l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_4}{dy}.\frac{dN_1}{dy} dydx = \int_0^l \int_0^1 \left(\frac{2x}{l^2} - \frac{1}{l^2} - \frac{x^2}{l^4}\right) dydx = 0.3l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_1}{dx}.\frac{dN_2}{dx} dxdy = \int_0^l \int_0^1 \left(\frac{2y}{l^3} - \frac{1}{l^2} - \frac{y^2}{l^4}\right) dxdy = -0.3l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_2}{dx}.\frac{dN_2}{dx} dxdy = \int_0^l \int_0^1 \left(\frac{1}{l^2} - \frac{2y}{l^3} - \frac{y^2}{l^4}\right) dxdy = 0.3l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_3}{dx}.\frac{dN_2}{dx} dxdy = \int_0^l \int_0^1 \left(\frac{y}{l^3} - \frac{y^2}{l^4}\right) dxdy = 0.2l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_4}{dx}.\frac{dN_2}{dx} dxdy = \int_0^l \int_0^1 \left(\frac{y}{l^3} - \frac{y^2}{l^4}\right) dxdy = 0.2l^{-1}$$

$$\int_0^l \int_0^1 \frac{dN_1}{dy}.\frac{dN_2}{dy} dydx = \int_0^l \int_0^1 \left(-\frac{x^2}{l^3} + \frac{x}{l^2}\right) dydx = 0.2$$

$$\int_0^l \int_0^1 \frac{dN_2}{dy}.\frac{dN_2}{dy} dydx = \int_0^l \int_0^1 \left(\frac{x^2}{l^2}\right) dydx = 0.3l$$

$$\int_0^l \int_0^1 \frac{dN_3}{dy}.\frac{dN_2}{dy} dydx = \int_0^l \int_0^1 \left(-\frac{x^2}{l^3}\right) dydx = -0.3$$

$$\int_0^l \int_0^1 \frac{dN_4}{dy}.\frac{dN_2}{dy} dydx = \int_0^l \int_0^1 \left(-\frac{x}{l^2} + \frac{x^2}{l^3}\right) dydx = -0.2$$

$$\int_0^l\int_0^1 \frac{dN_1}{dx}\cdot\frac{dN_3}{dx}dxdy = \int_0^l\int_0^1 \left(\frac{y^2}{l^4}-\frac{y}{l^3}\right)dxdy = -0.2l^{-1}$$
$$\int_0^l\int_0^1 \frac{dN_2}{dx}\cdot\frac{dN_3}{dx}dxdy = \int_0^l\int_0^1 \left(\frac{y}{l^3}-\frac{y^2}{l^4}\right)dxdy = 0.2l^{-1}$$
$$\int_0^l\int_0^1 \frac{dN_3}{dx}\cdot\frac{dN_3}{dx}dxdy = \int_0^l\int_0^1 \left(\frac{y^2}{l^4}\right)dxdy = 0.3l^{-1}$$
$$\int_0^l\int_0^1 \frac{dN_4}{dx}\cdot\frac{dN_3}{dx}dxdy = \int_0^l\int_0^1 \left(\frac{y^2}{l^4}\right)dxdy = 0.3l^{-1}$$
$$\int_0^l\int_0^1 \frac{dN_1}{dy}\cdot\frac{dN_3}{dy}dydx = \int_0^l\int_0^1 \left(\frac{x^2}{l^4}-\frac{x}{l^3}\right)dydx = 0.2l^{-1}$$
$$\int_0^l\int_0^1 \frac{dN_2}{dy}\cdot\frac{dN_3}{dy}dydx = \int_0^l\int_0^1 \left(-\frac{x^2}{l^3}\right)dydx = -0.3$$
$$\int_0^l\int_0^1 \frac{dN_3}{dy}\cdot\frac{dN_3}{dy}dydx = \int_0^l\int_0^1 \left(\frac{x^2}{l^4}\right)dydx = 0.3l^{-1}$$
$$\int_0^l\int_0^1 \frac{dN_4}{dy}\cdot\frac{dN_3}{dy}dydx = \int_0^l\int_0^1 \left(\frac{x}{l^3}-\frac{x^2}{l^4}\right)dydx = 0.2l^{-1}$$
$$\int_0^l\int_0^1 \frac{dN_1}{dx}\cdot\frac{dN_4}{dx}dxdy = \int_0^l\int_0^1 \left(\frac{y^2}{l^4}-\frac{y}{l^3}\right)dxdy = 0.2l^{-1}$$
$$\int_0^l\int_0^1 \frac{dN_2}{dx}\cdot\frac{dN_4}{dx}dxdy = \int_0^l\int_0^1 \left(\frac{y}{l^3}-\frac{y^2}{y^4}\right)dxdy = 0.2l^{-1}$$
$$\int_0^l\int_0^1 \frac{dN_3}{dx}\cdot\frac{dN_4}{dx}dxdy = \int_0^l\int_0^1 \left(\frac{y^2}{l^4}\right)dxdy = 0.3l^{-1}$$
$$\int_0^l\int_0^1 \frac{dN_4}{dx}\cdot\frac{dN_4}{dx}dxdy = \int_0^l\int_0^1 \left(\frac{y^2}{l^4}\right)dxdy = 0.3l^{-1}$$
$$\int_0^l\int_0^1 \frac{dN_1}{dy}\cdot\frac{dN_4}{dy}dydx = \int_0^l\int_0^1 \left(\frac{2x}{l^3}-\frac{1}{l^4}-\frac{x^2}{l^3}\right)dydx = -0.3l^{-1}$$
$$\int_0^l\int_0^1 \frac{dN_2}{dy}\cdot\frac{dN_4}{dy}dydx = \int_0^l\int_0^1 \left(-\frac{x}{l^2}+\frac{x^2}{l^3}\right)dydx = -0.2$$
$$\int_0^l\int_0^1 \frac{dN_3}{dy}\cdot\frac{dN_4}{dy}dydx = \int_0^l\int_0^1 \left(\frac{x}{l^3}-\frac{x^2}{l^4}\right)dydx = 0.2l^{-1}$$
$$\int_0^l\int_0^1 \frac{dN_4}{dy}\cdot\frac{dN_4}{dy}dydx = \int_0^l\int_0^1 \left(\frac{1}{l^2}-\frac{2x}{l^3}+\frac{x^2}{l^4}\right)dydx = 0.3l^{-1}$$

So that,

$$\int_0^l\int_0^1 [B]^T\,[B]dxdy = \begin{bmatrix} 0.3l^{-1} & 0 & -0.3l^{-1} & 0 & -0.2l^{-1} & 0 & 0.2l^{-1} & 0 \\ 0 & 0.3l^{-1} & 0 & 0.2 & 0 & 0.2l^{-1} & 0 & -0.3l^{-1} \\ 0.3l^{-1} & 0 & 0.3l^{-1} & 0 & 0.2l^{-1} & 0 & 0.2l^{-1} & 0 \\ 0 & 0.2 & 0 & 0.3l^{-1} & 0 & -0.3 & 0 & -0.2 \\ -0.2l^{-1} & 0 & 0.2l^{-1} & 0 & 0.3l^{-1} & 0 & 0.3l^{-1} & 0 \\ 0 & 0.2l^{-1} & 0 & -0.3 & 0 & 0.3l^{-1} & 0 & 0.2l^{-1} \\ -0.2l^{-1} & 0 & 0.2l^{-1} & 0 & 0.3l^{-1} & 0 & 0.3l^{-1} & 0 \\ 0 & 0.3l^{-1} & 0 & -0.2 & 0 & 0.2l^{-1} & 0 & 0.3l^{-1} \end{bmatrix}$$

$$AE\int_0^l\int_0^1 [B]^T\,[B]dA = \frac{AE}{l}\begin{bmatrix} 0.3 & 0 & -0.3 & 0 & -0.2 & 0 & 0.2 & 0 \\ 0 & 0.3 & 0 & 0.2l & 0 & 0.2 & 0 & -0.3 \\ 0.3 & 0 & 0.3 & 0 & 0.2 & 0 & 0.2 & 0 \\ 0 & 0.2l & 0 & 0.3 & 0 & -0.3l & 0 & -0.2l \\ -0.2 & 0 & 0.2 & 0 & 0.3 & 0 & 0.3 & 0 \\ 0 & 0.2 & 0 & -0.3l & 0 & 0.3 & 0 & 0.2 \\ -0.2 & 0 & 0.2 & 0 & 0.3 & 0 & 0.3 & 0 \\ 0 & 0.3 & 0 & -0.2l & 0 & 0.2 & 0 & 0.3 \end{bmatrix}$$

Also, specified in the form of 'q' per unit length be address in similar manner by performing the integration $\int [N]^T\,qdx$ and obtain $\{f\}$; thus, for a uniformly distributed force q, we can write the equivalent nodal force vector as:

$$\{f\}_C = \int_0^l \int_0^1 [N_i]^T q dxdy = q \begin{bmatrix} \frac{4l-3}{4} \\ \frac{4l-3}{4} \\ \frac{2l-1}{4} \\ \frac{2l-1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

Now for body forces or gravity loading which is a typical body force is given byρg per-volume or ρAg per- unit length, where ρ is the mass density of the continuum. The equivalent nodal force vector for the distributed body force can be obtained as:

$$\{f\} = \int_v [N_i]^T l\rho g dxdy = \int_0^l \int_0^1 [N_i]^T l\rho g dxdy$$

That is;

$$\{f\}_B = \int_0^l \int_0^1 [N_i]^T l\rho g dxdy = l\rho g \begin{bmatrix} \frac{4l-3}{4} \\ \frac{4l-3}{4} \\ \frac{2l-1}{4} \\ \frac{2l-1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix}$$

Hence, the numerical equivalence of equation (49); that is, $[k]\{\psi\} = \{f\}$ on a unit cell bases is:

$$\frac{AE}{l} \begin{bmatrix} 0.3 & 0 & -0.3 & 0 & -0.2 & 0 & 0.2 & 0 \\ 0 & 0.3 & 0 & 0.2l & 0 & 0.2 & 0 & -0.3 \\ 0.3 & 0 & 0.3 & 0 & 0.2 & 0 & 0.2 & 0 \\ 0 & 0.2l & 0 & 0.3 & 0 & -0.3l & 0 & -0.2l \\ -0.2 & 0 & 0.2 & 0 & 0.3 & 0 & 0.3 & 0 \\ 0 & 0.2 & 0 & -0.3l & 0 & 0.3 & 0 & 0.2 \\ -0.2 & 0 & 0.2 & 0 & 0.3 & 0 & 0.3 & 0 \\ 0 & 0.3 & 0 & -0.2l & 0 & 0.2 & 0 & 0.3 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = l\rho g \begin{bmatrix} \frac{4l-3}{4} \\ \frac{4l-3}{4} \\ \frac{2l-1}{4} \\ \frac{2l-1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} + q \begin{bmatrix} \frac{4l-3}{4} \\ \frac{4l-3}{4} \\ \frac{2l-1}{4} \\ \frac{2l-1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \\ \frac{1}{4} \end{bmatrix} \quad (69)$$

Where, the first term in equation (69) represents the stiffness matrix of the solid, the second term represents the axial and transverse displacement vector, the third term represents body forces, and the fifth term represents concentrated load or live load.

VI. Conclusion

Despite the discretization of a solid into finite element mesh. The field variables like displacement, stress, and strain can only be determine by numerical computation from a single finite element mesh. This complex engineering problem is first formulated as a functional, whose partial differential equivalence is obtained by a clumsy mathematical process. Then, the approximate numerical solution of the differential equation is sort using shape functions. All these analysis is done with a single tetra-shaped mesh; then, the assemblage into a system level is carried-out for the whole structure. Hence, any structure of any size can be analyze for its field variable as demonstrated here. Equation (69) is the numeral representation of Laplace equation. Simplifying the partial differential equation into a linear equation at unit level, whose assemblage, will give a system of equation representing the whole; making it, easier for the numerical determination of displacement if parameters like: Young modulus (E), Areas (A), Density (ρ), Length (l), and the concentrated load (q) are given.

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