Analyzing the multiplicative properties of numerical digits using diagonal arithmetic

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Abstract
Diagonal arithmetic as a new field springing up in mathematics no doubt possesses inestimable powers yet to be fully grasped by the human mind. When numbers are engaged in the operation of multiplication, the digits tend to exhibit certain properties that, if studied with adequate attention, could be put systematically in the form of sequences as if they had been naturally programmed to do so. This paper examines the multiplicative properties of digits such as 5, 6, 7, 8 and 9 with a view to establishing generating sequences having the potential to serve as alternatives for multiplying numbers.

Keywords: Analysis, multiplicative properties, numerical digits, diagonal arithmetic

1. Introduction

The concept of diagonal arithmetic was pioneered by Inah (2022), a Nigerian mathematician and mathematics educator of the 21st century. It has its roots from the concept of diagonal matrix in which non-zero entries exist only in the leading diagonal of a matrix. The following are examples of diagonal matrices.

$$\begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix} \qquad \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix} \qquad \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \end{pmatrix}$$

Inah (2022) proposed that all digits in any number having one or more digits can be represented as n x n diagonal matrix with the digits appearing in the leading diagonal of the matrix. According to this proposal, every number having one or more digits can be written in the form of a square matrix such that the entries in the leading diagonal are exactly the digits of the number in the order of their powers, while the entries outside the leading diagonal are all 0s. Thus by this statement we have the following

$$43 = \begin{matrix} 4 & 0 \\ & 3 \end{matrix} = \begin{pmatrix} 4 & 0 \\ 0 & 3 \end{pmatrix}$$

$$216 = \begin{matrix} 2 & 0 & 0 \\ & 1 & 0 \\ & & 6 \end{matrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 6 \end{pmatrix}$$

For convenience, it is proper to discard the 0s and to use a new symbol such as a left-sided brace { instead of brackets or parentheses ().

Therefore

$$43 = \left\{ \begin{array}{ll} 4 & \\ & 3 \end{array} \right. \qquad 216 = \left\{ \begin{array}{lll} 2 & & \\ & 1 & \\ & & 6 \end{array} \right.$$

One very important thing to note about this representation is that it allows for ease of renaming. This can be done in two directions: upward renaming and/or downward renaming. For example, renaming can be done on the number 216 in two ways: Downward renaming.

$$216 = \left\{ \begin{array}{lll} 2 & & \\ & 1 & \\ & & 6 \end{array} \right. = \left\{ \begin{array}{lll} {}^{1}1 & & \\ & 1 & \\ & & 6 \end{array} \right. = \left\{ \begin{array}{lll} 1 & & \\ & 11 & \\ & & 6 \end{array} \right.$$

$$216 = \left\{ \begin{array}{lll} 2 & & \\ & 1 & \\ & & 6 \end{array} \right. = \left\{ \begin{array}{lll} 2 & & \\ & {}^{1}0 & \\ & & 6 \end{array} \right. = \left\{ \begin{array}{lll} 1 & & \\ & 0 & \\ & & 16 \end{array} \right.$$

Upward renaming

$$\left\{ \begin{array}{lll} 1 & & \\ & 11 & \\ & & 6 \end{array} \right. = \left\{ \begin{array}{lll} {}^{1}1 & & \\ & 1 & \\ & & 6 \end{array} \right. = \left\{ \begin{array}{lll} 2 & & \\ & 1 & \\ & & 6 \end{array} \right. = 216$$

$$\left\{ \begin{array}{lll} 1 & & \\ & 0 & \\ & & 16 \end{array} \right. = \left\{ \begin{array}{lll} 2 & & \\ & {}^{1}0 & \\ & & 6 \end{array} \right. = \left\{ \begin{array}{lll} 2 & & \\ & 1 & \\ & & 6 \end{array} \right. = 216$$

The first two examples illustrate just how renaming of numbers can done in the downward direction from left to right. The last two examples illustrate renaming in the upward direction from right to left. Although this seems to be irrelevant, if considered carefully it could be seen to be the key behind the success of diagonal arithmetic.

Let us now try to see how the foregoing assertion can be true by retrospectively reflecting on the efforts of past mathematicians towards describing the properties of numbers. In this, it can be said that for centuries now such efforts have been more general than specific. By this is meant that mathematicians have been focusing more on the properties of numbers considered as 'wholes' rather than as 'digits' making up these 'wholes'. No doubt, when numbers are viewed as 'wholes', they exhibit certain general properties and we have seen several examples in number theory. Mathematicians do not seem to know much about the specific properties of the digits that make up the numbers and this limits our understanding of mathematics in some way. It becomes pertinent to ask whether or not the digits themselves possess any unique characteristics that can be generalized.

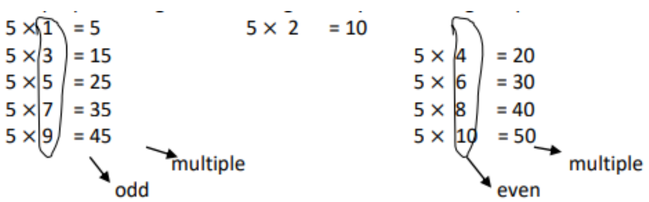
Indeed, of greater concern in this study are the multiplicative properties of the digits. This actually raises the following questions:

1. Is the deployment of diagonal arithmetic helpful in the analyses of the multiplicative- properties of numerical digits such as 5, 6, 7, 8, and 9?
2. Will diagonal arithmetic help in finding generating sequences for these digits?
3. How can such analysis be useful to the mathematics learner?

This paper seeks to provide answers to these pertinent questions by demonstrating the possibilities using diagonal arithmetic as a mathematical tool for analysis.

2. Multiplicative Properties of the digit 5

It is proper to begin this investigation by considering the products shown below



By observing the first arrangement where 5 multiplies odd numbers such as 1, 3, 5, 7 and 9, it must have been noticed that the multiples generated end with the digit 5. But is that the only information we derive from this observation? Perhaps there is more to learn from here. It is again wise to look at the table provided below.

Table 1: Generating sequence for digits in products of 5 by odd numbers

Product	Multiple	T	U	T	U
5 × 1	5	0	5	$\frac{1-1}{2}$	5
5 × 3	15	1	5	$\frac{3-1}{2}$	5
5 × 5	25	2	5	$\frac{5-1}{2}$	5
5 × 7	35	3	5	$\frac{7-1}{2}$	5
5 × 9	45	4	5	$\frac{9-1}{2}$	5

By observation, it can be seen that the tens digit is obtained by removing 1 from the odd number and then dividing by 2. If we let n = odd number, then the digit in the tens place is generated by the sequence $\frac{n-1}{2}$. Thus by this, we can conclude the following: If n = odd, then

$$\begin{array}{ll} & \text{T} \quad \text{U} \\ 5 \times n = & \frac{n-1}{2} \quad 5 \end{array} \tag{1}$$

In brace matrix form, equation (1) can be written as follows

$$5 \times n = \begin{cases} \frac{n-1}{2} \\ \quad 5 \end{cases} \tag{2}$$

Where n = 1, 3, 5, 7,…

Again, a closer look at the second arrangement in the first illustration will reveal that when 5 is multiplied by an even number, the multiple obtained has 0 in the unit place. However, we also observe that the tens digit is simply one-half of that particular even number multiplying 5. See the table below.

Table 2: Generating sequence for digits in products of 5 by even numbers

Product	Multiple	T	U	T	U
5 × 2	10	1	0	$\frac{2}{2}$	0
5 × 4	20	2	0	$\frac{4}{2}$	0
5 × 6	30	3	0	$\frac{6}{2}$	0
5 × 8	40	4	0	$\frac{8}{2}$	0
5 × 10	50	5	0	$\frac{10}{2}$	0

Thus if we let n to be that particular even number, then

$$5 \times n = \begin{cases} \frac{n}{2} \\ \quad 0 \end{cases} \tag{3}$$

Where n = 2, 4, 6,…

2.1. Multiplicative properties of the digit 6

Consider again the products shown below

6 × 1 = 6		6 × 2 = 12	
6 × 3 = 18			6 × 4 = 24
6 × 5 = 30			6 × 6 = 36
6 × 7 = 42			6 × 8 = 48
6 × 9 = 54			6 × 10 = 60
multiple odd		multiple even	

A closer look at the first arrangement where 6 is multiplied by an odd number will reveal something rather interesting. However, this cannot be seen clearly unless we tabulate the results as shown below.

Table 3: Generating sequence for digits in products of 6 by odd numbers

Product	Multiple	T	U	T	U
6 × 1	6	0	6	$\frac{1-1}{2}$	6 + 1 – 1
6 × 3	18	1	8	$\frac{3-1}{2}$	6 + 3 – 1
6 × 5	30	2	10	$\frac{5-1}{2}$	6 + 5 - 1
6 × 7	42	3	12	$\frac{7-1}{2}$	6 + 7 – 1
6 × 9	54	4	14	$\frac{9-1}{2}$	6 + 9 – 1

From table3 above, it can be seen that for any odd number, n, multiplying 6, the generating sequence for the unit digit is given by 6 + n – 1, while the generating sequence for the tens digit is given by $\frac{n-1}{2}$. Thus we have the following equations for the generating sequences of digits formed by 6

$$6 \times n = \begin{cases} \frac{n-1}{2} \\ 6 + n - 1 \end{cases} \qquad (4)$$

Where $n = 1, 3, 5, \ldots$

Note here that this is all very easy to understand if we can write the multiples in such a way that they are pictured as entries in a diagonal brace matrices, i.e,

$$6 \times 1 = 6 = \begin{cases} 0 \\ \quad 6 \end{cases} = \begin{cases} \frac{1-1}{2} \\ \quad 6 + 1 - 1 \end{cases}$$

$$6 \times 3 = 18 = \begin{cases} 1 \\ \quad 8 \end{cases} = \begin{cases} \frac{3-1}{2} \\ \quad 6 + 3 - 1 \end{cases}$$

$$6 \times 5 = 30 = \begin{cases} 3 \\ \quad 0 \end{cases} = \begin{cases} 2 \\ \quad 10 \end{cases} = \begin{cases} \frac{5-1}{2} \\ \quad 6 + 5 - 1 \end{cases}$$

$$6 \times 7 = 42 = \begin{cases} 4 \\ \quad 2 \end{cases} = \begin{cases} 3 \\ \quad 12 \end{cases} = \begin{cases} \frac{7-1}{2} \\ \quad 6 + 7 - 1 \end{cases}$$

$$6 \times 9 = 54 = \begin{cases} 5 \\ \quad 4 \end{cases} = \begin{cases} 4 \\ \quad 14 \end{cases} = \begin{cases} \frac{9-1}{2} \\ \quad 6 + 9 - 1 \end{cases}$$

Using the same procedure for the second arrangement in the second illustration, we have the following

$$6 \times 2 = 12 = \begin{cases} 1 & \\ & 2 \end{cases} = \begin{cases} \frac{2}{2} & \\ & 2 \end{cases}$$

$$6 \times 4 = 24 = \begin{cases} 2 & \\ & 4 \end{cases} = \begin{cases} \frac{4}{2} & \\ & 4 \end{cases}$$

$$6 \times 6 = 36 = \begin{cases} 3 & \\ & 6 \end{cases} = \begin{cases} \frac{6}{2} & \\ & 6 \end{cases}$$

$$6 \times 8 = 48 = \begin{cases} 4 & \\ & 8 \end{cases} = \begin{cases} \frac{8}{2} & \\ & 8 \end{cases}$$

$$6 \times 10 = 60 = \begin{cases} 6 & \\ & 0 \end{cases} = \begin{cases} \frac{10}{2} & \\ & 10 \end{cases}$$

Thus if n = even, then

$$6 \times n = \begin{cases} \frac{n}{2} & \\ & n \end{cases} \qquad (5)$$

2.2. Multiplicative properties of the digit 7

Following the previous steps, consider the products by 7 as shown below.

$7 \times 1 = 7$	$7 \times 2 = 14$
$7 \times 3 = 21$	$7 \times 4 = 28$
$7 \times 5 = 35$	$7 \times 6 = 42$
$7 \times 7 = 49$	$7 \times 8 = 56$
$7 \times 9 = 6$	$7 \times 10 = 70$
odd ↘ → multiple	even ↘ → multiple

Looking at the above, it is difficult to find connections anywhere. However, if we write them in diagonal brace matrix form, the results could be astonishing. Starting with the first arrangement, we have the following.

$$7 \times 1 = 7 = \begin{cases} 0 & \\ & 7 \end{cases} = \begin{cases} \frac{1-1}{2} & \\ & 7 + 2(1-1) \end{cases}$$

$$7 \times 3 = 21 = \begin{cases} 2 & \\ & 1 \end{cases} = \begin{cases} 1 & \\ & 11 \end{cases} = \begin{cases} \frac{3-1}{2} & \\ & 7 + 2(3-1) \end{cases}$$

$$7 \times 5 = 35 = \begin{cases} 3 & \\ & 5 \end{cases} = \begin{cases} 2 & \\ & 15 \end{cases} = \begin{cases} \frac{5-1}{2} & \\ & 7 + 2(5-1) \end{cases}$$

$$7 \times 7 = 49 = \begin{cases} 4 & \\ & 9 \end{cases} = \begin{cases} 3 & \\ & 19 \end{cases} = \begin{cases} \frac{7-1}{2} & \\ & 7 + 2(7-1) \end{cases}$$

$$7 \times 9 = 63 = \begin{cases} 6 & \\ & 3 \end{cases} = \begin{cases} 4 & \\ & 23 \end{cases} = \begin{cases} \frac{9-1}{2} & \\ & 7 + 2(9-1) \end{cases}$$

From the above, we can conclude that for some number, n,

$$7 \times n = \begin{cases} \frac{1-1}{2} & \\ & 7 + 2(1-1) \end{cases}$$

(6)

$if\ n = odd$

With the second arrangement, we have the following brace matrix forms

$$7 \times 2 = 14 = \begin{cases} 1 & \\ & 4 \end{cases} = \begin{cases} \frac{2}{2} & \\ & 2 \times 2 \end{cases}$$

$$7 \times 4 = 28 = \begin{cases} 2 & \\ & 8 \end{cases} = \begin{cases} \frac{4}{2} & \\ & 2 \times 4 \end{cases}$$

$$7 \times 6 = 42 = \begin{cases} 4 & \\ & 2 \end{cases} = \begin{cases} 3 & \\ & 12 \end{cases} = \begin{cases} \frac{6}{2} & \\ & 2 \times 6 \end{cases}$$

$$7 \times 8 = 56 = \begin{cases} 5 & \\ & 6 \end{cases} = \begin{cases} 4 & \\ & 16 \end{cases} = \begin{cases} \frac{8}{2} & \\ & 2 \times 8 \end{cases}$$

$$7 \times 10 = 70 = \begin{cases} 7 & \\ & 0 \end{cases} = \begin{cases} 5 & \\ & 20 \end{cases} = \begin{cases} \frac{10}{2} & \\ & 2 \times 10 \end{cases}$$

Looking at the above, is easy to see that for some number, n, we have

$$7 \times n = \begin{cases} \frac{n}{2} & \\ & 2n \end{cases}$$
$$if\ n = even$$

2.3. Multiplicative properties of the digit 8

Going as we did previously, let us consider the following products

8×1	$= 8$	8×2	$= 16$
8×3	$= 24$	8×4	$= 32$
8×5	$= 40$	8×6	$= 48$
8×7	$= 56$	8×8	$= 64$
8×9	$= 72$	8×10	$= 80$
↘ odd	→ multiple	↘ even	→ multiple

By attempting to make a re-arrangement of the first set of products with odd numbers, we get the following diagonal forms

$$8 \times 1 = 8 = \begin{cases} 0 & \\ & 8 \end{cases} = \begin{cases} \frac{1-1}{2} & \\ & 8 + 3(1-1) \end{cases}$$

$$8 \times 3 = 24 = \begin{cases} 2 & \\ & 4 \end{cases} = \begin{cases} 1 & \\ & 14 \end{cases} = \begin{cases} \frac{3-1}{2} & \\ & 8 + 3(3-1) \end{cases}$$

$$8 \times 5 = 40 = \begin{cases} 4 & \\ & 0 \end{cases} = \begin{cases} 2 & \\ & 20 \end{cases} = \begin{cases} \frac{5-1}{2} & \\ & 8 + 3(5-1) \end{cases}$$

$$8 \times 7 = 56 = \begin{cases} 5 & \\ & 6 \end{cases} = \begin{cases} 3 & \\ & 26 \end{cases} = \begin{cases} \frac{7-1}{2} & \\ & 8 + 3(7-1) \end{cases}$$

$$8 \times 9 = 72 = \begin{cases} 7 & \\ & 2 \end{cases} = \begin{cases} 4 & \\ & 32 \end{cases} = \begin{cases} \frac{9-1}{2} & \\ & 8 + 3(9-1) \end{cases}$$

From the foregoing, it can be seen that the tens digit is generated by the sequence $\frac{n-1}{2}$, for some n which is odd, while the unit digit is generated by the sequence $8 + 3(n - 1)$. Thus the mathematical representation of the sequence generating digits for multiples of 8 is given by the equation.

$$8 \times n = \begin{cases} \frac{n-1}{2} & \\ & 8+3(n-1) \end{cases} \quad (8)$$

Where $n = odd$

By attempting to re-arrange the second arrangement with even numbers, we have the following diagonal forms.

$$8 \times 2 = 16 = \begin{cases} 1 & \\ & 6 \end{cases} = \begin{cases} \frac{2}{2} & \\ & 3 \times 2 \end{cases}$$

$$8 \times 4 = 32 = \begin{cases} 3 & \\ & 2 \end{cases} = \begin{cases} 2 & \\ & 12 \end{cases} = \begin{cases} \frac{4}{2} & \\ & 3 \times 4 \end{cases}$$

$$8 \times 6 = 48 = \begin{cases} 4 & \\ & 8 \end{cases} = \begin{cases} 3 & \\ & 18 \end{cases} = \begin{cases} \frac{6}{2} & \\ & 3 \times 6 \end{cases}$$

$$8 \times 8 = 64 = \begin{cases} 6 & \\ & 4 \end{cases} = \begin{cases} 4 & \\ & 24 \end{cases} = \begin{cases} \frac{8}{2} & \\ & 3 \times 8 \end{cases}$$

$$8 \times 10 = 80 = \begin{cases} 8 & \\ & 0 \end{cases} = \begin{cases} 5 & \\ & 30 \end{cases} = \begin{cases} \frac{10}{2} & \\ & 3 \times 10 \end{cases}$$

By observing the above closely, it can easily be seen that the tens digit is generated by the sequence $\frac{n}{2}$ for some n = even, while the unit digit is generated by the sequence 3 n. Thus the generating sequence for multiples of 8 is given by

$$8 \times n = \begin{cases} \frac{n}{2} & \\ & 3n \end{cases} \quad (9)$$

Where $n = even$

2.9. Multiplicative properties of the digit 9

Finally, let us consider the product of numbers by 9 as shown below

9 × 1 = 9	9 × 2 = 18
9 × 3 = 27	9 × 4 = 36
9 × 5 = 45	9 × 6 = 54
9 × 7 = 63	9 × 8 = 72
9 × 9 = 81	9 × 10 = 90
↘ odd → multiple	↘ even → multiple

If we re-arrange or re-write the first arrangement in diagonal brace matrix form, we get the following.

$$9 \times 1 = 9 = \begin{cases} 0 & \\ & 9 \end{cases} = \begin{cases} \frac{1-1}{2} & \\ & 9+4(1-1) \end{cases}$$

$$9 \times 3 = 27 = \begin{cases} 2 & \\ & 7 \end{cases} = \begin{cases} 1 & \\ & 17 \end{cases} = \begin{cases} \frac{3-1}{2} & \\ & 9+4(3-1) \end{cases}$$

$$9 \times 5 = 45 = \begin{cases} 4 & \\ & 5 \end{cases} = \begin{cases} 2 & \\ & 25 \end{cases} = \begin{cases} \frac{5-1}{2} & \\ & 9+4(5-1) \end{cases}$$

$$9 \times 7 = 63 = \begin{cases} 6 & \\ & 3 \end{cases} = \begin{cases} 3 & \\ & 33 \end{cases} = \begin{cases} \frac{7-1}{2} & \\ & 9+4(7-1) \end{cases}$$

$$9 \times 9 = 81 = \begin{cases} 8 & \\ & 1 \end{cases} = \begin{cases} 4 & \\ & 41 \end{cases} = \begin{cases} \frac{9-1}{2} & \\ & 9+4(9-1) \end{cases}$$

It can be seen here too, that for some odd number, the generating sequence for the tens digit in the multiples of 9 is given by $\frac{n-1}{2}$, while the generating sequence for the unit digit is given by $9 + 4(n - 1)$ for $n = odd$. Thus the equation representing the generating function of digits for multiples of 9 is given by

$$9 \times n = \begin{cases} \frac{n-1}{2} \\ 9+4(n-1) \end{cases} \quad (10)$$

$for\ n = 1, 3, 5, \ldots$

In the same vein, if we re-arrange the second arrangement in the first illustration under this section, we get the following.

$$9 \times 2 = 18 = \begin{cases} 1 \\ 8 \end{cases} = \begin{cases} \frac{2}{2} \\ 4 \times 2 \end{cases}$$

$$9 \times 4 = 36 = \begin{cases} 3 \\ 6 \end{cases} = \begin{cases} 2 \\ 16 \end{cases} = \begin{cases} \frac{4}{2} \\ 4 \times 4 \end{cases}$$

$$9 \times 6 = 54 = \begin{cases} 5 \\ 4 \end{cases} = \begin{cases} 3 \\ 24 \end{cases} = \begin{cases} \frac{6}{2} \\ 4 \times 6 \end{cases}$$

$$9 \times 8 = 72 = \begin{cases} 7 \\ 2 \end{cases} = \begin{cases} 4 \\ 32 \end{cases} = \begin{cases} \frac{8}{2} \\ 4 \times 8 \end{cases}$$

$$9 \times 10 = 90 = \begin{cases} 9 \\ 0 \end{cases} = \begin{cases} 5 \\ 40 \end{cases} = \begin{cases} \frac{10}{2} \\ 4 \times 10 \end{cases}$$

Here again, we can see that the tens digit is generated by the sequence $\frac{n}{2}$, for $n = even$, while the unit digit is generated by the sequence $4n$. Thus the equation of the generating sequence for multiples of 9 is given by

$$9 \times n = \begin{cases} \frac{n}{2} \\ 4n \end{cases} \quad (11)$$

$n = 2, 4, 6, \ldots$

3. Results and discussion

The first objective of the study was to determine if the deployment of diagonal arithmetic is helpful in the analysis of the multiplicative properties of numerical digits. Indeed, this goal has been satisfactorily met. We have already seen in this paper that numerical digits possess certain properties that can be harnessed into what this study identifies as generating sequences. This can be easily seen only if we try to write these multiples in brace matrix form, a concept borrowed from diagonal arithmetic. The second objective of this paper was to ascertain the possibility of finding generating sequences multiples of 5, 6, 7, 8, and 9. This goal was divided into four sections. First, we had to show that if 5 multiples any number, the generating sequence for the digits in the multiples of 5 is given by the equation

$$5 \times n = \begin{cases} \frac{n-1}{2} \\ 5 \end{cases} \quad (12)$$

$if\ n = odd$

And

$$5 \times n = \begin{cases} \frac{n}{2} \\ 0 \end{cases} \quad (13)$$

$if\ n = even$

The relations in equations (2) and (3) are true for all positive values of n. They are also true for all negative values of n provided that the sign does not interfere with the operation of multiplication. For example, we can see from the illustration below.

Alternative method

$$5 \times 25 = \left\{\begin{matrix} \frac{25-1}{2} & \\ & 5 \end{matrix}\right. = \left\{\begin{matrix} \frac{24}{2} & \\ & 5 \end{matrix}\right. = \left\{\begin{matrix} 12 & \\ & 5 \end{matrix}\right. = \left\{\begin{matrix} 1 & & \\ & 2 & \\ & & 5 \\ \hline \underline{1} & \underline{2} & \underline{5} \end{matrix}\right.$$

$$5 \times -73 = -\left\{\begin{matrix} \frac{73-1}{2} & \\ & 5 \end{matrix}\right. = -\left\{\begin{matrix} \frac{72}{2} & \\ & 5 \end{matrix}\right. = -\left\{\begin{matrix} 36 & \\ & 5 \end{matrix}\right. = -\left\{\begin{matrix} 3 & & \\ & 6 & \\ & & 5 \\ \hline -\underline{3} & \underline{6} & \underline{5} \end{matrix}\right.$$

$$5 \times 16 = \left\{\begin{matrix} \frac{16}{2} & \\ & 0 \end{matrix}\right. = \left\{\begin{matrix} 8 & \\ & 0 \\ \hline \underline{8} & \underline{0} \end{matrix}\right.$$

$$5 \times -24 = -\left\{\begin{matrix} \frac{24}{2} & \\ & 0 \end{matrix}\right. = \left\{\begin{matrix} 12 & \\ & 0 \end{matrix}\right. = \left\{\begin{matrix} 1 & & \\ & 2 & \\ & & 0 \\ \hline \underline{1} & \underline{2} & \underline{0} \end{matrix}\right.$$

The foregoing example shows that the negative sign should always be set aside so that it does not interfere with the final answer. Quite obviously, this kind of multiplication provides another useful alternative to diagonal multiplication which has not been known before this time. Here we learn that even in multiplication, division is inherent and it provides another route to the answer. For instance, going by the conventional process, we have the following.

Usual method

$$5 \times 25 = 5 \times \left\{\begin{matrix} 2 & \\ & 5 \end{matrix}\right. = \left\{\begin{matrix} 5 \times 2 & \\ & 5 \times 5 \end{matrix}\right. = \left\{\begin{matrix} 10 & \\ & 25 \end{matrix}\right. = \left\{\begin{matrix} 1 & & \\ & {}^{2}0 & \\ & & 5 \\ \hline \underline{1} & \underline{2} & \underline{5} \end{matrix}\right.$$

$$5 \times -73 = 5 \times -\left\{\begin{matrix} 7 & \\ & 3 \end{matrix}\right. = -\left\{\begin{matrix} 5 \times 7 & \\ & 5 \times 3 \end{matrix}\right. = -\left\{\begin{matrix} 35 & \\ & 15 \end{matrix}\right. = -\left\{\begin{matrix} 3 & & \\ & {}^{1}5 & \\ & & 5 \\ \hline -\underline{3} & \underline{6} & \underline{5} \end{matrix}\right.$$

$$5 \times 16 = 5 \times \left\{\begin{matrix} 1 & \\ & 6 \end{matrix}\right. = \left\{\begin{matrix} 5 \times 1 & \\ & 5 \times 6 \end{matrix}\right. = \left\{\begin{matrix} 5 & \\ & 30 \end{matrix}\right. = \left\{\begin{matrix} {}^{3}5 & \\ & 0 \\ \hline \underline{8} & \underline{0} \end{matrix}\right.$$

$$5 \times -24 = 5 \times -\left\{\begin{matrix} 2 & \\ & 4 \end{matrix}\right. = -\left\{\begin{matrix} 5 \times 2 & \\ & 5 \times 4 \end{matrix}\right. = -\left\{\begin{matrix} 10 & \\ & 20 \end{matrix}\right. = -\left\{\begin{matrix} 1 & & \\ & {}^{2}0 & \\ & & 5 \\ \hline -\underline{1} & \underline{2} & \underline{0} \end{matrix}\right.$$

Both methods can be used as alternatives to the conventional process of multiplication. The beauty about these two methods in diagonal arithmetic is that renaming is easily understood because it is clear enough.
Second, we had to show that the multiplicative properties of 6 as a digit when multiplied by any number is given by

$$6 \times n = \left\{\begin{matrix} \frac{n-1}{2} & \\ & 6 + n - 1 \end{matrix}\right. \tag{14}$$

$if\ n = odd$
And

$$6 \times n = \left\{\begin{matrix} \frac{n}{2} & \\ & n \end{matrix}\right. \tag{15}$$

$if\ n = even$

Equations (4) and (5) are both true for all positive and negative values of n. They also provide useful alternatives to diagonal multiplication by 6. We can compare between the following methods.

Alternative method

$$6 \times 17 = \begin{cases} \frac{17-1}{2} \\ 6 + 17 - 1 \end{cases} =$$

$$\begin{cases} \frac{n}{2} \\ 22 \end{cases} = \begin{cases} 8 \\ 22 \end{cases} = \begin{cases} ^{2}8 \\ 2 \end{cases} \quad \begin{cases} 10 \\ 2 \end{cases} = \begin{cases} 1 \\ 0 \\ 2 \end{cases} \quad \underline{1 \quad 0 \quad 2}$$

$$6 \times 18 = \begin{cases} \frac{18}{2} \\ 18 \end{cases} = \begin{cases} 9 \\ 18 \end{cases} = \begin{cases} ^{1}9 \\ 8 \end{cases} = \begin{cases} 10 \\ 8 \end{cases} = \begin{cases} 1 \\ 0 \\ 8 \end{cases} \quad \underline{1 \quad 0 \quad 8}$$

Usual method

$$6 \times 17 = 6 \times \begin{cases} 1 \\ 7 \end{cases} = \begin{cases} 6 \times 1 \\ 6 \times 7 \end{cases} = \begin{cases} 6 \\ 42 \end{cases} = \begin{cases} ^{4}6 \\ 2 \end{cases} = \begin{cases} 10 \\ 2 \end{cases}$$

$$= \begin{cases} 1 \\ 0 \\ 2 \end{cases} \quad \underline{1 \quad 0 \quad 2}$$

$$6 \times 18 = 6 \times \begin{cases} 1 \\ 8 \end{cases} = \begin{cases} 6 \times 1 \\ 6 \times 8 \end{cases} = \begin{cases} 6 \\ 48 \end{cases} = \begin{cases} ^{4}6 \\ 8 \end{cases} = \begin{cases} 10 \\ 8 \end{cases}$$

$$= \begin{cases} 1 \\ 0 \\ 8 \end{cases} \quad \underline{1 \quad 0 \quad 8}$$

Third, we had to show that for any number multiplying 7, we have the following generating sequences for the digits.

$$7 \times n = \begin{cases} \frac{1-1}{2} \\ 7 + 2(1-1) \end{cases} \quad \text{if } n = odd \tag{16}$$

And

$$7 \times n = \begin{cases} \frac{n}{2} \\ 2n \end{cases} \quad \text{if } n = even \tag{17}$$

Both equations (6) and (7) are true for all positive and negative values of n provided the negative sign does not interfere with the operation. They also provide useful alternatives to diagonal multiplication by 7 as shown below.

Alternative method

$$7 \times 25 = \begin{cases} \frac{25-1}{2} \\ 7 + 2(25-1) \end{cases} = \begin{cases} \frac{24}{2} \\ 7 + 2(24) \end{cases}$$

$$= \begin{cases} 12 \\ 55 \end{cases} = \begin{cases} 1 \\ ^{5}2 \\ 5 \end{cases} \quad \underline{1 \quad 7 \quad 5}$$

$$7 \times 28 = \begin{cases} \frac{28}{2} \\ 2(28) \end{cases} = \begin{cases} 14 \\ 56 \end{cases} = \begin{cases} 1 \\ ^{5}4 \\ 6 \end{cases} \quad \underline{1 \quad 9 \quad 6}$$

Usual method

$$7 \times 25 = 7 \times \begin{cases} 2 \\ \quad 5 \end{cases} = \begin{cases} 7 \times 2 \\ \quad 7 \times 5 \end{cases} = \begin{cases} 14 \\ \quad 35 \end{cases} = \begin{cases} 1 \quad {}^{3}4 \\ \qquad\quad \underline{5} \\ \underline{\underline{1 \quad 7 \quad 5}} \end{cases}$$

$$7 \times 28 = 7 \times \begin{cases} 2 \\ \quad 8 \end{cases} = \begin{cases} 7 \times 2 \\ \quad 7 \times 8 \end{cases} = \begin{cases} 14 \\ \quad 56 \end{cases} = \begin{cases} 1 \quad {}^{5}4 \\ \qquad\quad \underline{6} \\ \underline{\underline{1 \quad 9 \quad 6}} \end{cases}$$

Fourth, we had to show that for some number, n, multiplied by 8, the generating sequences for the digits is given by the following equations

$$8 \times n = \begin{cases} \frac{n-1}{2} \\ \quad 8 + 3(n-1) \end{cases} \tag{18}$$

$where\ n = odd$

And

$$8 \times n = \begin{cases} \frac{n}{2} \\ \quad 3n \end{cases} \tag{19}$$

$where\ n = even$

Both equations (8) and (9) are true for all values of, n and they also serve as alternatives to diagonal multiplication. The following examples show that this is true.

Alternative method

$$8 \times 15 = \begin{cases} \frac{15-1}{2} \\ \quad 8 + 3(15-1) \end{cases} = \begin{cases} \frac{14}{2} \\ \quad 8 + 3(14) \end{cases} = \begin{cases} 7 \\ \quad 50 \end{cases}$$

$$= \begin{cases} {}^{5}7 \\ \quad 0 \end{cases} = \begin{cases} 12 \\ \quad 0 \end{cases} = \begin{cases} 1 \quad 2 \\ \qquad \underline{0} \\ \underline{\underline{1 \quad 2 \quad 0}} \end{cases}$$

$$8 \times 24 = \begin{cases} \frac{24}{2} \\ \quad 3(24) \end{cases} = \begin{cases} 12 \\ \quad 72 \end{cases} = \begin{cases} 1 \quad {}^{7}2 \\ \qquad\quad \underline{1} \\ \underline{\underline{1 \quad 9 \quad 2}} \end{cases}$$

Usual method

$$8 \times 15 = 8 \times \begin{cases} 1 \\ \quad 5 \end{cases} = \begin{cases} 8 \times 1 \\ \quad 8 \times 5 \end{cases} = \begin{cases} 8 \\ \quad 40 \end{cases}$$

$$= \begin{cases} {}^{4}8 \\ \quad 0 \end{cases} = \begin{cases} 12 \\ \quad 0 \end{cases} = \begin{cases} 1 \quad 2 \\ \qquad \underline{0} \\ \underline{\underline{1 \quad 2 \quad 0}} \end{cases}$$

$$8 \times 24 = 8 \times \begin{cases} 2 \\ \quad 4 \end{cases} = \begin{cases} 8 \times 2 \\ \quad 8 \times 4 \end{cases} = \begin{cases} 16 \\ \quad 32 \end{cases} = \begin{cases} 1 \quad {}^{3}6 \\ \qquad\quad \underline{2} \\ \underline{\underline{1 \quad 9 \quad 2}} \end{cases}$$

Finally, we had to show that for all positive and negative values of n,

$$9 \times n = \left\{\begin{array}{ll} \frac{n-1}{2} & \\ & 9+4(n-1) \end{array}\right. \qquad (20)$$

$if\ n = odd$

And

$$9 \times n = \left\{\begin{array}{ll} \frac{n}{2} & \\ & 4n \end{array}\right. \qquad (21)$$

$if\ n = even$

Both equations (9) and (10) are true for all n and they can serve as alternatives to usual diagonal multiplication. The following examples will illustrate their applicability.

Alternative method

$$9 \times 47 = \left\{\begin{array}{ll} \frac{47-1}{2} & \\ & 9+4(47-1) \end{array}\right. = \left\{\begin{array}{ll} \frac{46}{2} & \\ & 9+4(46) \end{array}\right. = \left\{\begin{array}{ll} 23 & \\ & 193 \end{array}\right.$$

$$= \left\{\begin{array}{lll} {}^{1}2 & & \\ & {}^{9}3 & \\ & & 3 \end{array}\right. = \left\{\begin{array}{lll} 3 & & \\ & 12 & \\ & & 3 \end{array}\right. = \left\{\begin{array}{lll} {}^{1}3 & & \\ & 2 & \\ & & 3 \\ \hline \underline{4} & \underline{2} & \underline{3} \end{array}\right.$$

$$9 \times 48 = \left\{\begin{array}{ll} \frac{48}{2} & \\ & 4(48) \end{array}\right. = \left\{\begin{array}{ll} 24 & \\ & 192 \end{array}\right. = \left\{\begin{array}{lll} {}^{1}2 & & \\ & {}^{9}4 & \\ & & 2 \end{array}\right. = \left\{\begin{array}{lll} 3 & & \\ & 13 & \\ & & 2 \end{array}\right.$$

$$= \left\{\begin{array}{lll} {}^{1}3 & & \\ & 3 & \\ & & 2 \\ \hline \underline{4} & \underline{3} & \underline{2} \end{array}\right.$$

Usual method

$$9 \times 47 = 9 \times \left\{\begin{array}{ll} 4 & \\ & 7 \end{array}\right. = \left\{\begin{array}{ll} 9 \times 4 & \\ & 9 \times 7 \end{array}\right. = \left\{\begin{array}{ll} 36 & \\ & 63 \end{array}\right. = \left\{\begin{array}{lll} 3 & & \\ & {}^{6}6 & \\ & & 3 \end{array}\right.$$

$$= \left\{\begin{array}{lll} 3 & & \\ & 12 & \\ & & 3 \end{array}\right. = \left\{\begin{array}{lll} {}^{1}3 & & \\ & 2 & \\ & & 3 \\ \hline \underline{4} & \underline{2} & \underline{3} \end{array}\right.$$

$$9 \times 48 = 9 \times \left\{\begin{array}{ll} 4 & \\ & 8 \end{array}\right. = \left\{\begin{array}{ll} 9 \times 4 & \\ & 9 \times 8 \end{array}\right. = \left\{\begin{array}{ll} 36 & \\ & 72 \end{array}\right. = \left\{\begin{array}{lll} 3 & & \\ & {}^{7}6 & \\ & & 2 \end{array}\right.$$

$$= \left\{\begin{array}{lll} 3 & & \\ & 13 & \\ & & 2 \end{array}\right. = \left\{\begin{array}{lll} {}^{1}3 & & \\ & 3 & \\ & & 2 \\ \hline \underline{4} & \underline{2} & \underline{2} \end{array}\right.$$

4. Conclusion

Diagonal arithmetic, no doubt, holds a huge potential for revealing something beyond the ordinary eye. In this particular case, it has been successfully utilized to analyze the multiplicative properties of numerical figures constituting the set, {5, 6, 7, 8, and 9}. Based on the foregoing analyses and expositions, therefore, the following conclusions can be adduced.

All digits of numbers multiplied by 5, 6, 7, 8, and/or 9 behave in certain unique way that can be summarized into a generating sequence for finding their products;

For any number, n, multiplied by 5, 6, 7, 8, or 9, the generating sequence or relation for the digits is given by the general equation

$$k \times n = \left\{\begin{array}{ll} \frac{n-1}{2} & \\ & k+(k-5)(n-1) \end{array}\right. \qquad (22)$$

$if\ n = odd$

And $5 \leq k \leq 9$

Note here that equation (12) is the general formula for finding the generating sequence for the digits if n is odd.
3) For any number n multiplied by 5, 6, 7, 8, and 9, the generating sequence for the digits is given by the general equation

$$k \times n = \begin{cases} \frac{n}{2} \\ (k-5)n \end{cases} \tag{23}$$

$if\ n = even$
And $5 \leq k \leq 9$

Equation (13) is the general formula for finding the generating sequence for the digits if n is even.
4) Both equations (12) and (13) are true for all n, whether positive or negative.
5) All the generating sequences for the digits provide useful alternatives to diagonal multiplication and to the conventional process of multiplication. They may also have other useful applications but these are yet to be known.

5. References

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