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Reduction of the cubic equation to the depressed form by completing the cube

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Abstract
Solutions to cubic equation have been explored extensively by mathematicians over the centuries. Different solutions have been devised for cubic equations most of which are able to solve reducible cubic equations and they usually involve trial and error procedures. 16th century mathematicians also devised solutions for irreducible cubic equations. One way of solving irreducible cubic equation usually involves the substitution $x = y - \frac{b}{3a}$ which reduces the given equation to the depressed form, $y3 = py + q$. This actually raises the question as to how this substitution is obtained, and whether or not some deductive steps can prove it analytically. For long now, it has been inconceivable that the procedure deployed in the method of completing the square for quadratic equations can be extended to cubic equations. This study investigated the possibility of providing a general and more reliable method of finding the solutions to cubic equations by deploying the method of completing the cube. This method is not only an extension of the method of completing the square but also a proof of the substitution, $x = y - \frac{b}{3a}$. It $3a$ shows just how this substitution is generated.

Keywords: Reducible, irreducible, cubic, depressed, equation

1. Introduction

The problem of solving cubic equations of the form

$$ax^3+bx^2+cx+d = 0 \quad (1)$$

Has been explored extensively by mathematicians over the centuries. In this paper, however, the subject is given renewed interest with a view to establishing a new method of solution that can possibly eliminate trial and error. It has already been demonstrated that if the cubic function.

$$f(x)= ax^3+bx^e+cx+d \quad (2)$$

Is reducible to its linear forms i.e., if it can be expressed as a product of linear factors such that

$$f(x)= (a1x+a0)(b1x+b0)(c1x+c0) \quad (3)$$

Where a1, a0, b1, b0, c1, c0 are all integers, then one of the solutions can be found by trial and error. By applying the factor theorem the other two solutions are easily obtained. Generally, these three solutions are known as 'the roots' of the cubic equation (1).

The procedure is the same if the function (2) is reducible to the form

$$f(x)= (a2x^2+a1x +a0) (b1x + b0) \quad (4)$$

Where one of the factors is an irreducible quadratic expression. In this case, the distinct solution is first found by trial and error. Then the irreducible quadratic equation is solved by completing the square method if its roots are real but not distinct.

Various root-testing techniques have been devised over the centuries. However, they all usually involve trial and error approaches. The problem with these trial and error approaches is that it can be very difficult at times, especially if the coefficients of the terms are real but indistinct. This actually poses a great deal of challenge during problem solving.

Moreover, if the cubic equation has real but indistinct roots, then the cubic function (2) is completely irreducible. In this case, the trial and error methods will fail completely. To get around this problem, 16thcentury mathematicians were able to demonstrate that since all cubic equations of the form (1) have at least one real root, applying the substituting $x = y - \frac{b}{3a}$ will reduce equation (1) to the depressed cubic equation

$$y^3 = py + q \quad (5).$$

The solution of this depressed cubic equation (5) has already been worked out by these set of mathematicians. However, only one root of the depressed real root is obtained by the method devised by Tart glia and Cardano. After obtaining the real root of the depressed cubic equation, we can find one real root for the cubic equation (1) and then use the factor theorem to find the other two roots.

The foregoing actually raises the following questions

1. Is it possible to extend the procedure adopted in solving quadratic equations by completing the square to cubic equations?
2. Is there any analytical proof of the substitution? $x = y - \frac{b}{3a}$?
3. Is it possible to find the solution to reducible cubic equations without resulting to trial and error methods as well as the factor theorem?
4. Is it possible to reduce the cubic equation (1) to the depressed form (5) by avoiding the substitution? $x=y - \frac{b}{3a}$?

This paper seeks to answer the afore-stated questions by demonstrating the possibilities in them.

2. Extending the concept of completing the square to cubic equation

As stated already, the concept of completing the cube is built upon the framework of completing the square, which is applied on quadratic equations. By carefully observing the development of the concept of completing the square, it is not difficult to see the connections and to extend this to higher degree equations.

This section shall be divided into two: first, it will be demonstrating the possibility to make a cubic expression a perfect cube just the way a quadratic expression is made a perfect square; second, it will be shown that it is possible to complete the cube on the cubic equation in a similar fashion, as did the quadratic equation.

2.1. Making a cubic expression a perfect cube

Consider the following expansion

$(x + 1)^2 = x^2 + 2(1)x + 1^2$

$(x + 2)^2 = x^2 + 2(2)x + 2^2$

$(x + 3)^2 = x^2 + 2(3)x + 3^2$

$(x + 4)^2 = x^2 + 2(4)x + 4^2$

$(x + n)^2 = x^2 + 2(n)x + n^2$

Now, to complete the square on the quadratic expression $x^2 + \mu x$, we ask; "what must be added to $x^2 + \mu x$ to make it a perfect square?"

If we let n^2 be the number to be added then

$x^2 + \mu x + n^2 = (x + n)^2$

$x^2 + \mu x + n^2 = x^2 + 2(n)x + n^2$

By comparing terms, we see that

$2n = \mu$

$\therefore n = \frac{\mu}{2\alpha}$

Thus we have $x^2 + \mu x + (\frac{\mu}{2})^2 = x + \frac{\mu}{2}$

Suppose that we have an expression of the form $x^2 + \frac{\beta}{\alpha}x$, what must we add to it to make it a perfect square? By the same procedure,

$$x^2 + \frac{\beta}{\alpha}x = (x + n)^2$$

$$\rightarrow x^2 + \frac{\beta}{\alpha}x + n^2 = x^2 + 2nx + n^2$$

Comparing terms we have

$2n = \frac{\beta}{\alpha}$

$\therefore\ n = \frac{\beta}{2\alpha}$

Thus $x^2 + \frac{\beta}{\alpha}x + (\frac{\beta}{2\alpha})^2 = (x + \frac{\beta}{2\alpha})^2$ (6)

Similarly, if we examine closely the following expansions:
$(x + 1)^3 = x^3 + 3x^2 + 3(1)x + 1^3$
$(x + 2)^3 = x^3 + 3(2)x^2 + 3(2)^2x + 2^3$
$(x + 3)^3 = x^3 + 3(3)x^2 + 3(3)^2x + 3^3$
$(x + 4)^3 = x^3 + 3(4)x^2 + 3(4)^2x + 4^3$
$(x + n)^3 = x^3 + 3(n)x^2 + 3(n)^2x + n^3$
$= x^3 + 3n\,x^2 + 3n^2x + n^3$

To complete the cube on the cubic expression $x^3 + ax^2$, we ask, what must be added to $x^3 + ax^2$ to make it a perfect cube?" Now, if we let
$3k^2x + k^3$ be the expression to be added, then
$x^3 + ax^2 + 3k^2x + k^3 = (x + k)^3$
$\rightarrow x^3 + ax^2 + 3k^2x + k^3 = x^3 + 3kx^2 + 3k^2x + k^3$
by comparing terms, we get
$3k = a$
$\therefore\ k = \frac{a}{3}$

Suppose that we have an expression of the form $x^3 + \frac{b}{a}x^2$, what must we add?
By repeating the same procedure as before, we get

$x^3 + \frac{b}{a}x^2 + 3k^2x + k^3 = (x + k)^3$

$\rightarrow x^3 + \frac{b}{a}x^2 + 3k^2x + k^3 = x^3 + 3kx^2 + 3k^2x + k^3$

Comparing terms again, we get

$3k = \frac{b}{a}$

$k = \frac{b}{3a}$ (7)

Thus $x^3 + \frac{b}{a}x^2 + 3(\frac{b}{3a})^2x + (\frac{b}{3a})^3 = (x + \frac{b}{3a})^3.$

2.2. Completing the cube on the cubic equation
To complete the square on the quadratic equation

$\alpha x^2 + \beta x + \gamma = 0$ (8)

Rewriting (8) in the form $x^2 + \frac{\beta}{\alpha}x = \frac{-\gamma}{\alpha}$

Adding n^2 to both sides gives $x^2 + \frac{\beta}{\alpha}x + n^2 = \frac{-\gamma}{\alpha} + n^2$

$\rightarrow (x + n)^2 = \frac{-\gamma}{\alpha} + n^2$

From (6), $n = \frac{\beta}{2\alpha}$, substituting this gives

$(x + \frac{\beta}{2\alpha})^2 = \frac{-\gamma}{\alpha} + (\frac{\beta}{2\alpha})^2 = \frac{-\gamma}{\alpha} + \frac{\beta^2}{4\alpha^2} = -\ \frac{4\alpha\gamma + \beta^2}{4\alpha^2}$

$$:.\ (x+\frac{\beta}{2\alpha})^2 = \frac{4^2-4\alpha\gamma}{4\alpha^2} \quad (9)$$

Similarly, to complete the cube on the cubic equation (1)

$$ax^3 + bx^2 + cx + d = 0$$

Re-writing (1) into the form $x^3 + \frac{a}{b}x^2 = \frac{-c}{\alpha}x - \frac{d}{\alpha}$

Completing the cube on the left, we get

$$x^3 + \frac{a}{b}x^2 + 3k^2x + k^3 = \frac{-c}{\alpha}x + 3k^2x - \frac{d}{\alpha} + k^3$$

$$\rightarrow \quad (x + k)^3 = (\frac{-c}{a} + 3k^2)x\frac{-d}{a} + k^3$$

From (7), $k = \frac{b}{3\alpha}$, substituting this into the above gives

$$(x + \frac{b}{3\alpha})^3 = \left[\frac{-c}{a} + 3(\frac{b}{3\alpha})^2\right]x - \frac{d}{\alpha} + (\frac{b}{3\alpha})^2$$

$$= \left(\frac{-c}{\alpha} + \frac{3b^2}{9a^2}\right)x - \left(\frac{d}{\alpha} - \frac{b^3}{27a^3}\right)$$

$$= \left(\frac{-c}{\alpha} + \frac{b^2}{3a^2}\right)x - \left(\frac{27a^2d - b^2}{27a^3}\right)$$

$$= \left(\frac{-3ac + b^2}{3a^2}\right)x + \left(\frac{b^2 - 27a^2d}{27a^3}\right)$$

$$:.(x + \frac{b}{3\alpha})^3 = \left(\frac{b^2 - 3ac}{3a^2}\right)x + \left(\frac{b^3 - 27a^2d}{27a^3}\right) \quad (10)$$

2.3. Proof of the substitution $x = y - \frac{b}{3\alpha}$

From the result obtained in equation (10), it is easy to see that if we let

$$y = x + \frac{b}{3\alpha}$$

$$\text{Then, } x = y - \frac{b}{3\alpha} \quad (11)$$

Equation (11) is the substitution made intuitively by 16th century mathematicians such as Tartaglia and Cardano. It is also clear from here that equation (11) is obtained by completing the cube on equation (1). Hence, it is safe to say that equation (11) has been proven analytically by the method of completing the cube.

2.4. Solution to the reducible equation

From the equation (10), we have

$$(x + \frac{b}{3\alpha})^3 = \left(\frac{b^2 - 3ac}{3a^2}\right)x + \left(\frac{b^3 - 27a^2d}{271^3}\right)$$

$$\rightarrow (x + \frac{b}{3\alpha})^3 = \frac{b^2 - 3ac}{3a^2}\left[x + \frac{b^3 - 27a^2d}{9a(b^2 - 3ac)}\right]$$

If $\frac{b^3 - 27a^2d}{9a(b^2 - 3ac)} = \frac{b}{3\alpha}$

Then $x = \frac{b}{3\alpha}$ is a solution of the cubic equation (1) which in this case will have three distinct real roots.

$\rightarrow b^3 - 271a^2d = 3b^2 - 9abc$

$\rightarrow b^3 - 3b^3 - 27a^2d + 9abc = 0$

$$9abc - 2b^3 - 27a^2d = 0 \quad (12)$$

Equation (12) is the condition for $x = \frac{b}{3a}$ to be a solution to the cubic equation

(1). If this condition holds then

$(x + \frac{b}{3a})^3 = \frac{b^2-3ac}{3a^2}(x + \frac{b}{3a})$

$\rightarrow (x + \frac{b}{3a})^3 - \left(\frac{b^2-3ac}{3a^2}\right)(x + \frac{b}{3a}) = 0$

$\rightarrow (x + \frac{b}{3a})\left[(x + \frac{b}{3a})^2 - \left(\frac{b^2-3ac}{3a^2}\right)\right] = 0$

$\rightarrow \left(x + \frac{b}{3a}\right)\left(x + \frac{b}{3a} - \sqrt{\frac{b^2-3ac}{3a^2}}\right)\left(x + \frac{b}{3a}\sqrt{\frac{b^2-3ac}{3a^2}}\right) = 0$

$$\rightarrow \left(x + \frac{b}{3a}\right)\left(x + \frac{b}{3a} - \frac{1}{a}\sqrt{\frac{b^2-3ac}{3}}\right)\left(x + \frac{b}{3a} + \frac{1}{a}\sqrt{\frac{b^2-3ac}{3}}\right) = 0 \quad (13)$$

Equation (13) reveals the three distinct roots of the equation (1) if $x = \frac{b}{3a}$ is also a solution to the same equation. If, however, $x \neq \frac{b}{3a}$, the equation (10) will need to be transformed into the depressed cubic equation by using the substitutions in (11) if trial and error procedures must be avoided.

2.5 Reducing the cubic equation to its depressed form

Equation (10) provides that

$$\left(x + \frac{b}{3a}\right)^3 = \left(\frac{b^2-3ac}{3a^2}\right)x + \left(\frac{b^3-27a^2d}{27a^3}\right)$$

If $b^2 - 3ac = 0$, then

$$\left(x + \frac{b}{3a}\right)^3 = \frac{b^3-27a^2d}{27a^3}$$

$$\rightarrow \quad x + \frac{b}{3a} = \sqrt[3]{\frac{b^3-27a^2d}{27a^3}} = \sqrt[3]{\frac{b^3-27a^2d}{3a}}$$

$$\rightarrow \quad x = \frac{-b}{3a} + \left(\frac{\sqrt[3]{b^3-27a^2d}}{3a}\right)$$

$$\therefore \quad x = \frac{-b + \sqrt[3]{b^3-27a^2d}}{3a} \quad (14)$$

If b2 – 3ac ≠ 0 and b3 – 27a2 d = 0, then

$$\left(x + \frac{b}{3a}\right)^3 = \left(\frac{b^2-3ac}{3a^2}\right)x$$

Substituting equation (11) into the above we get

$$y^3 = \frac{b^2-3ac}{3a^2}\left(y - \frac{b}{3a}\right)$$

$$\rightarrow \quad y^3 = \left(\frac{b^2-3ac}{3a^2}\right)y - \frac{b}{3a}\left(\frac{b^2-3ac}{3a^2}\right)$$

$$\rightarrow \quad y^3 = \left(\frac{b^2-3ac}{3a^2}\right)y + \frac{b^3+3abc}{9a^3}$$

$$:. \quad y^3 = \left(\frac{b^2-3ac}{3a^2}\right)y + \frac{3abc-b^3}{9a^3}$$

$$:. \quad y^3 = py + t \qquad (15)$$

Where $p = \frac{b^2-3abc}{9a^2}$

And $t = \frac{3abc-b^3}{9a^3}$

If also $3abc - b^3 = 0$, then equation (15) reduces to

$y^3 + py$

$\rightarrow y(y^2 + p) = 0$

$:.\ y = 0$ or $y^2 + p = 0$

$\rightarrow\ y = 0$ or $y = \pm\sqrt{-p}$

This requires that $p < 0$ if the above must hold.

Again if

$b^2 - 3ac \neq 0$ and $b^3 - 27a^2 d \neq 0$ then by substituting (11) into (10) we get

$$y^3 = \left(\frac{b^2-3ac}{3a^2}\right)\left(y - \frac{b}{3a}\right) + \left(\frac{b^3-27a^2d}{27a^3}\right)$$

$$y^3 = \left(\frac{b^2-3ac}{3a^2}\right)y - \frac{b}{3a}\left(\frac{b^2-3ac}{3a^2}\right) + \left(\frac{b^3-27a^2d}{27a^3}\right)$$

$$y^3 = \left(\frac{b^2-3ac}{3a^2}\right)y + \frac{-b^3+3abc}{9a^3} + \frac{b^3-27a^2d}{27a^3}$$

$$\rightarrow y^3 = \left(\frac{b^2-3ac}{3a^2}\right)y + \frac{-3b^3+9abc+b^3-27a^2d}{27a^3}$$

$$\rightarrow y^3 = \left(\frac{b^2-3ac}{3a^2}\right)y + \frac{9abc-2b^3-27a^2d}{27a^3}$$

$$:. \quad y^3 = py + q \qquad (16)$$

Where $p = \frac{b^2-3ac}{3a^2}$ and $q = \frac{9abc+2b^3-27a^2d}{27a^3}$

Equation (15) and (16) are known as depressed cubic equations. The solution to this depressed cubic equation can be found by using the substitution $y = u + v$.

Similarly, if we let

$$\mu = x + \frac{\beta}{2\alpha},$$

then $$x = \mu - \frac{\beta}{2\alpha}$$

Substituting these into equation (8), we get

$$\mu^2 = \frac{\beta^2 - 4\,\alpha\,8}{4\alpha^2},$$

then we get

$\mu^2 = \gamma$ (17)

Equation (17) is the depressed quadratic equation which of cause is easily solves. However, equation (16) requires more procedures to be adopted to find its solution. Besides, equation (17) has two solutions while equation (16) has three solutions one of which must be real.

Note

1. Equation (1) is depressed by completing the cube to obtain equation (16) This depression eliminates the term that is just one degree less than the term with the highest degree.
2. Equation (8) is depressed by completing the square to obtain equation (17). This depression eliminates the term that is just one degree less than the term with the highest degree.

The foregoing, therefore, is the case whenever the method of completing the square, cubic, quartic, etc. is adopted.

3. Results and discussion

By making the quadratic expression $x^2 + \mu x$a perfect square, we find that

$\left(\frac{\mu}{2}\right)^2$ must be added

i.e.$x^2 + \mu x + \left(\frac{\mu}{2}\right)^2 = \left(x + \frac{\mu}{2}\right)^2$.

Similarly, by making the cubic expression$x^3 + \frac{b}{a}x^2$ a perfect cube, we find that

we must add $3\left(\frac{b}{3a}\right)^2 x + \left(\frac{b}{3a}\right)^3$

i.e. $x^3 + \frac{b}{a}x^2 + 3\left(\frac{b}{3a}\right)^2 x + \left(\frac{b}{3a}\right)^3 = \left(x + \frac{b}{3a}\right)^3$.

Again, if we complete the square on the quadratic equation (8)

$\alpha x^2 + \beta x + \gamma = 0$

We obtain the result in equation (9)

$$\left(x + \frac{\beta}{2\alpha}\right)^2 = \frac{\beta^2 - 4\alpha 8}{4\alpha^2}$$

Similarly by completing the cube on the equation (1)

$ax^3 + bx^2 + cx + d = 0$

We obtain the result in equation (10)

$$\left(x + \frac{b}{3a}\right)^3 = \left(\frac{b^2 - 3ac}{3a^2}\right)x + \left(\frac{b^3 - 27a^2 d}{27a^3}\right).$$

These results show that it is possible to extend the concept of completing the square applied on quadratic equations to cubic equations. In this case, it will be to complete the cube on the cubic equation.

From equation (9), letting

$$\mu = x + \frac{\beta}{2\alpha}, \text{ then } x = \mu - \frac{\beta}{2\alpha}$$

Similarly, from equation (10), letting

$y = x + \frac{b}{3a}$, then $x = y - \frac{b}{3a}$.

These results show that the substitution $x = y - \frac{b}{3a}$, which transforms the cubic equation (1) into the depressed cubic equation (16) can be proven analytically by the method of completing the cube. This also shows that the transformation is not just an intuitive guess

Equation (10) can also be utilized to show that if

$9abc - 2b^3 - 27a^2 d = 0$

Then $x = \frac{b}{3a}$ is a solution to equation (1) which has three rootsat this point i.e.

$$\left(x + \frac{b}{3a}\right)\left(x + \frac{b}{3a} - \frac{1}{a}\sqrt{\frac{b^2 - 3ac}{3}}\right)\left(x + \frac{b}{3a} + \frac{1}{a}\sqrt{\frac{b^2 - 3ac}{3}}\right) = 0$$

If $x = \frac{b}{3a}$ is not a root of the equation (1), then the above does not hold. The next step will be to reduce the equation to its depressed form by the transformation $x = y - \frac{b}{3a}$ and $y = x + \frac{b}{3a}$. This shows that only when $x = \frac{b}{3a}$ is a solution to the cubic equation will it be possible to reduce equation (1) to its linear forms directly by the method of completing the cube. This is sadly the case even if equation (1) is reducible. If the above does not hold, then by using the transformation in equation (11), we obtain the depressed cubic

$y^3 = py + q$

Which can be solved using the substitution y = u + v to obtain one solution for y. This value can then be used to obtain x. The other x values are found through the usual factor theorem. This shows that equation (16) is obtained by the method of completing the cube and actually by the transformation $x = y - \frac{b}{3a}$. Indeed, one can say $x = y - \frac{b}{3a}$ is not actually a transformation but a result obtained by completing the cube on the cubic equation (1). It goes further to illustrate that the method of completing the cube actually eliminates the problem of solving cubic equations by trial and error procedures, which can sometime be challenging to execute.

4. Conclusion

Based on the results obtained so far, the following conclusions can be adduced

1. It is possible to ext end the procedures adopted in solving quadratic equations by the method of completing the square to cubic equations. In this case, it can be called ‘completing the cube’.
2. The method of completing the cube is an analytical proof for the transformation x = y - b $3a$ which was used intuitively by 16th century mathematicians to solve cubic equations.
3. The method of completing the cube can be used to solve reducible equation without resorting to trial and error procedures.
4. The method of completing the cube can be used holistically to solve all types of cubic equations because it transforms the cubic equation (1) to its depressed form which can be solved easily. This is true irrespective of the reducibility of the equation.

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