



# International Journal of Multidisciplinary Research and Growth Evaluation.

## Improved L-Stable method for solving initial value problem

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### Article Info

**ISSN (online):** 2582-7138

**Volume:** 03

**Issue:** 01

January-February 2022

**Received:** 02-01-2022;

**Accepted:** 18-01-2022

**Page No:** 393-400

### Abstract

In this manuscript, a fifth order convergent one-step explicit method having L-stability is developed to deal with non-autonomous and autonomous initial value problems of ordinary differential equations (ODEs). The method is applied to many initial value problems and found more efficient and accurate, than existing methods of identical types. The proposed method is found to be a good solving technique for initial value problems having singular solution, stiff problems and singularly perturbed.

**Keywords:** Local truncation error, Absolute error, L-stability, autonomous and non-autonomous

### 1. Introduction

Solving ordinary differential equations is of great worth among scientist, researchers, and mathematicians due to their significant applicability in modeling a physical phenomenon and scientific problems. Several numerical techniques are widely used to solve differential equations arising from different fields of engineering, science, chemical kinetics, population model, physics, and electrical networks, that are difficult or could not be solved analytically <sup>[1-4]</sup>. The presence of a pole in the solution or discontinuity in terms containing lower order derivative are specified as singular initial value problem. Generally, Taylor method and Runge-kutta type methods and some linear multistep methods usually failed or very poor in performance near singularities because they are based on polynomial approximation. While the performance of some methods based upon rational approach is much better when solution passes through the singularity <sup>[5-9]</sup>. A rational approximation whose denominator is of greater degree than its numerator produces an L-stable method. More often L-stable type methods are non-linear methods <sup>[1]</sup>. In this regard, in this paper related with the development of a fifth order non-linear method based on rational approximation, to deal with various types of initial value problems having singularities <sup>[11-15]</sup>. The proposed method is found to be L-stable and will be utilized for the numerical integration of the initial value problem represents (1). The method has been tested on a variety of IVPs of first order ODEs. The comparison among the proposed method with some existing methods determines that the proposed method gives more accurate outcomes as associated to them.

In next Section (i.e., section no. 2) the derivation of the L-stable method is given. The stability analysis is considered in Section 3 Local truncation error in section 4, error analysis is carried out in Section 5, finally, result discussion and conclusions put an end to the manuscript.

### 2. Derivation of proposed method

Consider the first-order initial value problem (IVP)

$$\frac{dy}{dt} = f(t, y), y(t_0) = y_0 \quad (1)$$

Where  $y, f(t, y) \in \mathbb{R}, t \in [a, b] \subseteq \mathbb{R}$ .

We assume that the problem has a single and continuously differentiable solution. In other words it is a well-posed problem. Here we express  $y_k \approx y(t_k)$ , where  $y_k$  is the approximate to the theoretical solution of  $y(t_k)$  at nodal point  $t_k = a + kh$ ;  $k = 0, 1, 2, \dots, K$  where  $h = \frac{b-a}{K}$  is step-size. After the study of [1-3] where a one-step explicit rational method is proposed to get an approximate solution of (1) resultantly, equation as follows to get fifth-order convergent method. It's foist that the numerous solution at  $t = t_{k+1}$  is specific by

$$y_{k+1} = \frac{\alpha_k + \beta_k h + \gamma_k h^2}{1 + \omega_k h + \psi_k h^2 + \eta_k h^3}, \quad (2)$$

Where  $\alpha_k, \beta_k, \gamma_k, \omega_k, \psi_k, \eta_k$  are unknown constants found with the help of Taylor series?

$$y_{k+1} = [\alpha_k + (\beta_k - \alpha_k \omega_k)h + (\gamma_k - \alpha_k \psi_k + (-\beta_k + \alpha_k \omega_k)\omega_k)h^2 + (-\alpha_k \eta_k + (-\beta_k + \alpha_k \omega_k)\psi_k + (-\gamma_k + \alpha_k \psi_k + \omega_k \beta_k - \alpha_k \omega_k^2)h^3 + ((-\beta_k + \alpha_k \omega_k)\eta_k + (-\gamma_k + \alpha_k \psi_k + \omega_k \beta_k - \alpha_k \psi_k^2)\psi_k + (\alpha_k \eta_k + \psi_k \beta_k - 2\varphi_k \alpha_k \omega_k + \omega_k \gamma_k - \omega_k^2 \beta_k + \alpha_k \omega_k^3)\omega_k)h^4 + ((-\gamma_k + \alpha_k \psi_k + \omega_k \beta_k - \alpha_k \omega_k^2)\eta_k + (\alpha_k \eta_k + \psi_k \beta_k - 2\psi_k \alpha_k \omega_k + \omega_k \gamma_k - \omega_k^2 \beta_k + \alpha_k^3)\psi_k + (\eta_k \beta_k - 2\eta_k \alpha_k \omega_k + \psi_k \gamma_k - \alpha_k \psi_k^2 - 2\psi_k \omega_k \beta_k + 3\psi_k \alpha_k \omega_k^2 - \omega_k^2 \gamma_k + \omega_k^3 \beta_k - \alpha_k \omega_k^4)\omega_k)h^5] + O(h^6).$$

Equating with Taylor's series of  $y_{k+1}$  about  $t = t_k$ . The constants  $\alpha_k, \beta_k, \gamma_k, \omega_k, \psi_k, \eta_k$  are determined by equating coefficients of above equation, with the coefficients of Taylor series up to  $h^5$ .

$$\alpha_k = y_k,$$

$$\begin{aligned} \beta_k &= \frac{1}{5} \left( \frac{-120(y'_k)^2 y''_k y'''_k + 30(y'_k)^3 y_k^{(iv)} + 40y'_k y_k (y_k''')^2 + 90y'_k (y_k'')^3 - 5y_k^2 y_k''' y_k^{(iv)} - 30y_k (y_k'')^2 y_k'''}{-6y_k^{(v)} (y'_k)^2 + 3y_k^{(v)} y_k^2 y_k''} \right. \\ &\quad \left. - 24y'_k y_k'' y_k''' - 3y_k'' y_k y_k^{(iv)} + 6(y'_k)^2 y_k^{(iv)} + 4y_k (y_k''')^2 + 18(y_k''')^2 \right), \\ \gamma_k &= -\frac{1}{20} \left( \frac{360y'_k y_k'' y_k''' + 60y_k'' y_k y_k^{(iv)} - 120y_k'' (y'_k)^2 y_k^{(iv)} - 80y_k'' y_k (y_k''')^2 - 180(y_k'')^4 - 5y_k^2 (y_k^{(iv)})^2}{-24y'_k y_k'' y_k''' - 3y_k'' y_k y_k^{(iv)} + 6(y'_k)^2 y_k^{(iv)} + 4y_k (y_k''')^2 + 18(y_k''')^2} \right. \\ &\quad \left. + 4y_k^2 y_k^{(v)} y_k''' - 24y_k'' y_k y_k^{(v)} + 40y_k y_k^{(iv)} y_k y_k''' - 80(y_k''')^2 (y'_k)^2 + 24y_k^{(v)} (y'_k)^2 \right), \\ \omega_k &= -\frac{1}{5} \left( \frac{15y'_k y_k'' y_k^{(iv)} - 5y_k y_k''' y_k^{(iv)} + 20y'_k (y_k''')^2 - 30(y_k'')^2 y_k''' - 6y_k^{(v)} (y'_k)^2 + 3y_k^{(v)} y_k (y'_k)^2}{-24y'_k y_k'' y_k''' - 3y_k'' y_k y_k^{(iv)} + 6(y'_k)^2 y_k^{(iv)} + 4y_k (y_k''')^2 + 18(y_k''')^2} \right), \\ \psi_k &= \frac{1}{20} \left( \frac{-5y_k (y_k^{(iv)})^2 + 4y_k y_k''' y_k^{(iv)} + 30y_k^{(iv)} (y_k'')^2 - 40(y_k'') (y_k''')^2 - 12y_k'' y_k' y_k^{(iv)} + 20y_k^{(iv)} y_k' y_k'''}{-24y'_k y_k'' y_k''' - 3y_k'' y_k y_k^{(iv)} + 6(y'_k)^2 y_k^{(iv)} + 4y_k (y_k''')^2 + 18(y_k''')^2} \right), \\ \eta_k &= \frac{1}{60} \left( \frac{60y_k''' y_k'' y_k^{(iv)} - 40(y_k''')^3 - 15y'_k (y_k^{(iv)})^2 + 12y'_k y_k''' y_k^{(iv)} - 18y_k^{(v)} (y_k'')^2}{-24y'_k y_k'' y_k''' - 3y_k'' y_k y_k^{(iv)} + 6(y'_k)^2 y_k^{(iv)} + 4y_k (y_k''')^2 + 18(y_k''')^2} \right), \\ y_{k+1} &= z \left( \begin{aligned} &y_n + \frac{1}{5} \left( \frac{-120(y'_k)^2 y''_k y'''_k + 30(y'_k)^3 y_k^{(iv)} + 40y'_k y_k (y_k''')^2 + 90y'_k (y_k'')^3 - 5y_k^2 y_k''' y_k^{(iv)} - 30y_k (y_k'')^2 y_k'''}{-6y_k^{(v)} (y'_k)^2 + 3y_k^{(v)} y_k^2 y_k''} \right) h \\ &- \frac{1}{20} \left( \frac{360y'_k y_k'' y_k''' + 60y_k'' y_k y_k^{(iv)} - 120y_k'' (y'_k)^2 y_k^{(iv)} - 80y_k'' y_k (y_k''')^2 - 180(y_k'')^4 - 5y_k^2 (y_k^{(iv)})^2}{-24y'_k y_k'' y_k''' - 3y_k'' y_k y_k^{(iv)} + 6(y'_k)^2 y_k^{(iv)} + 4y_k (y_k''')^2 + 18(y_k''')^2} \right) h^2 \\ &- \frac{1}{5} \left( \frac{15y'_k y_k'' y_k^{(iv)} - 5y_k y_k''' y_k^{(iv)} + 20y'_k (y_k''')^2 - 30(y_k'')^2 y_k''' - 6y_k^{(v)} (y'_k)^2 + 3y_k^{(v)} y_k (y'_k)^2}{-24y'_k y_k'' y_k''' - 3y_k'' y_k y_k^{(iv)} + 6(y'_k)^2 y_k^{(iv)} + 4y_k (y_k''')^2 + 18(y_k''')^2} \right) h \\ &+ \frac{1}{20} \left( \frac{-5y_k (y_k^{(iv)})^2 + 4y_k y_k''' y_k^{(iv)} + 30y_k^{(iv)} (y_k'')^2 - 40(y_k'') (y_k''')^2 - 12y_k'' y_k' y_k^{(iv)} + 20y_k^{(iv)} y_k' y_k'''}{-24y'_k y_k'' y_k''' - 3y_k'' y_k y_k^{(iv)} + 6(y'_k)^2 y_k^{(iv)} + 4y_k (y_k''')^2 + 18(y_k''')^2} \right) h^2 \\ &+ \frac{1}{60} \left( \frac{60y_k''' y_k'' y_k^{(iv)} - 40(y_k''')^3 - 15y'_k (y_k^{(iv)})^2 + 12y'_k y_k''' y_k^{(iv)} - 18y_k^{(v)} (y_k'')^2}{-24y'_k y_k'' y_k''' - 3y_k'' y_k y_k^{(iv)} + 6(y'_k)^2 y_k^{(iv)} + 4y_k (y_k''')^2 + 18(y_k''')^2} \right) h^3 \end{aligned} \right) \quad (3)$$

. After simplification, we get.

$$y_{k+1} = \frac{\begin{pmatrix} -3(480f_k^2 j_k k_k h + 120f_k^2 k_k h + 120f_k^3 l_k h - 24m_k f_k h^2 - 360f_k j_k^2 k_k h^2 + 180j_k^4 h^2 + 24j_k f_k m_k h^2 \\ + 80k_k^2 f_k^2 h^2 - 24m_k f_k^3 h^2) + (-480f_k j_k k_k + 120f_k^2 l_k + 360j_k^3 + 160f_k k_k^2 h - 120j_k^2 k_k h - 60j_k^2 l_k h^2 \\ + 80j_k k_k^2 h^2 - 90l_k k_k h^2) y_k + (60j_k l_k + 80k_k^2 - 20k_k l_k h + 12m_k j_k + 4h^2 m_k k_k) y_k \\ 1440f_k j_k k_k - 6f_k^3 l_k - 18j_k^3 + 240k_k^2 f_k - (180f_k j_k l_k + 360j_k^2 k_k + 6m_k f_k^2) h + \\ (90l_k j_k^2 - 120j_k k_k^2 - 36j_k f_k) h^2 + (60l_k f_k k_k) h^2 + (60k_k j_k l_k - 40k_k^3 - 15f_k l_k^2) + \\ 12f_k k_k m_k - 18m_k j_k^2) h^3 + (3j_k l_k - 4k_k^2 - (60k_k l_k - 3m_k j_k) h - (15l_k^2 - 12k_k l_k) h^2) y_k \end{pmatrix}}{\begin{pmatrix} 1440s'(t)s''(t)s'''(t)h + 120s'(t)^2 s'''(t)h + 120s'(t)^3 s^{(iv)}(t)h - 24s^{(v)}(t)s'(t)h^2 \\ - 360s'(t)s''(t)^2 s'''(t)h^2 + 180s''(t)^4 h^2 + 24s''(t)s'(t)s^{(iv)}(t)h^2 + 80s'''(t)^2 s'(t)^2 h^2 \\ - 24s'(t)^3 s^{(iv)}(t)h^2 + (-480s'(t)s''(t)s'''(t) + 120s'(t)^2 s^{(iv)}(t) + 360s''(t)^3 + 160s'''(t)^2 s'(t)h \\ - 120s''(t)^2 s'''(t)h - 60s''(t)s^{(iv)}(t)h^2 + 80s''(t)^2 s'''(t)h^2 - 90s'''(t)s^{(iv)}(t)h^2) y_k + \\ (60s''(t)s^{(iv)}(t) + 80s'''(t)^2 - 20s''(t)s^{(iv)}(t)h + 4s'''(t)s^{(iv)}(t)) y_k \\ 1440s'(t)s''(t)s'''(t)h - 6s'(t)^3 s^{(iv)}(t) - 18s''(t)^3 + 240s'''(t)^2 s'(t) - \\ (180s'(t)s''(t)s^{(iv)}(t) + 360s''(t)^2 s'''(t) + 6s^{(v)}(t)s''(t)^2)h + (90s^{(iv)}(t)s'''(t)^2 - 120s''(t)s'''(t)^2 \\ - 36s''(t)s'(t)h^2 + (60s'(t)s'''(t)s^{(iv)}(t)h^2 + (60s'''(t)s''(t)s^{(iv)}(t) - 40s'''(t)^3 - 15s'(t)s^{(iv)}(t)^2) \\ + 12s'(t)s'''(t)s^{(iv)}(t) - 18s^{(v)}(t)s''(t)^2)h^3 \\ (3s''(t)s^{(iv)}(t) - 4s'''(t)^2 - (60s'''(t)s^{(iv)}(t) - 3s^{(v)}(t)s''(t)h - (15s^{(iv)}(t)^2 - 12s^{(iv)}(t)s'''(t)h^2) y_k \end{pmatrix}}$$

Where  $y_k = y(t_k)$ ,  $y_{k+1} \approx y(t_{k+1})$ ,  $f_k = f(t_k, y_k)$  and,  $l_k = \frac{d^3 f}{dt^3}(t_k, y_k)$ ,

$k_k = \frac{d^2 f}{dt^2}(t_k, y_k)$ ,  $m_k = \frac{d^4 f}{dt^4}(t_k, y_k)$ ,

### 3. Local Truncation Error

Truncation error of the proposed method (3) can be obtained by considering the functional equation given below:

$$\mathcal{L}(s(t), h) = s(t+h) - \frac{\begin{pmatrix} -3(480s(t)^2 s''(t)s'''(t)h + 120s'(t)^2 s'''(t)h + 120s'(t)^3 s^{(iv)}(t)h - 24s^{(v)}(t)s'(t)h^2 \\ - 360s'(t)s''(t)^2 s'''(t)h^2 + 180s''(t)^4 h^2 + 24s''(t)s'(t)s^{(iv)}(t)h^2 + 80s'''(t)^2 s'(t)^2 h^2 \\ - 24s'(t)^3 s^{(iv)}(t)h^2 + (-480s'(t)s''(t)s'''(t) + 120s'(t)^2 s^{(iv)}(t) + 360s''(t)^3 + 160s'''(t)^2 s'(t)h \\ - 120s''(t)^2 s'''(t)h - 60s''(t)s^{(iv)}(t)h^2 + 80s''(t)^2 s'''(t)h^2 - 90s'''(t)s^{(iv)}(t)h^2) y_k + \\ (60s''(t)s^{(iv)}(t) + 80s'''(t)^2 - 20s''(t)s^{(iv)}(t)h + 4s'''(t)s^{(iv)}(t)) y_k \\ 1440s'(t)s''(t)s'''(t)h - 6s'(t)^3 s^{(iv)}(t) - 18s''(t)^3 + 240s'''(t)^2 s'(t) - \\ (180s'(t)s''(t)s^{(iv)}(t) + 360s''(t)^2 s'''(t) + 6s^{(v)}(t)s''(t)^2)h + (90s^{(iv)}(t)s'''(t)^2 - 120s''(t)s'''(t)^2 \\ - 36s''(t)s'(t)h^2 + (60s'(t)s'''(t)s^{(iv)}(t)h^2 + (60s'''(t)s''(t)s^{(iv)}(t) - 40s'''(t)^3 - 15s'(t)s^{(iv)}(t)^2) \\ + 12s'(t)s'''(t)s^{(iv)}(t) - 18s^{(v)}(t)s''(t)^2)h^3 \\ (3s''(t)s^{(iv)}(t) - 4s'''(t)^2 - (60s'''(t)s^{(iv)}(t) - 3s^{(v)}(t)s''(t)h - (15s^{(iv)}(t)^2 - 12s^{(iv)}(t)s'''(t)h^2) y_k \end{pmatrix}}$$

Where,  $s(t)$  is a function defined over interval of integration and differentiable  $n$  times? The local truncation error of the above method is obtained by the terms in power of  $h$  that are collected from Taylor series expanded around  $t$ ,

$$T_{k+1} = \frac{1}{7200} \left( \frac{1}{24y_k' y_k'' y_k''' + 3y_k'' y_k y_k^{(iv)} - 6(y_k')^2 y_k^{(iv)} - 4y_k(y_k''')^2 - 18(y_k'')^3} \right) [240y_k''' y_k'' y_k' y_k^{(vi)} + 30y_k^{(iv)} y_k'' y_k^{(vi)} - 60y_k^{(iv)} (y_k') y_k^{(vi)} - 40(y_k''')^2 y_k^{(vi)} y_k - 180y_k^{(iv)} (y_k'')^3 - 1800y_k^{(iv)} y_k'' y_k^{(vi)} + 800(y_k''')^4 + 600y_k''' y_k (y_k^{(iv)})^2 - 480(y_k'')^2 y_k' y_k^{(iv)} + 720(y_k'')^2 y_k' y_k'' - 75(y_k^{(iv)})^3 + 120y_k^{(iv)} y_k y_k''' y_k^{(v)} + 450(y_k^{(iv)})^2 (y_k')^2 - 360y_k^{(iv)} y_k' y_k^{(iv)} y_k' + 72(y_k^{(iv)})^2 (y_k')^2 - 36(y_k^{(iv)})^2 y_k] h^6 + O(h^7).$$

(4)

This confirms that the proposed method has fifth order of accuracy. The above obtained local truncation error of the method is of order sixth, where  $y_k', y_k'', y_k''', y_k^{(iv)}, y_k^{(v)}$  represent the values to the first, second, third, fourth, fifth derivatives of  $y(t)$  at point  $t$  respectively, provided that  $24y_k' y_k'' y_k''' + 3y_k'' y_k y_k^{(iv)} - 6(y_k')^2 y_k^{(iv)} - 4y_k(y_k''')^2 - 18(y_k'')^3 \neq 0$

Hence, whenever the solution of the differential equation in (1) is a function of the following form. The described proposed method is true:

$$y(t) = \frac{\alpha_k + \beta_k h + \gamma_k h^2}{\rho_k + \omega_k h + \psi_k h^2 + \eta_k h^3}, \quad (5)$$

Where  $\alpha_k, \beta_k, \gamma_k, \rho_k, \omega_k, \psi_k, \eta_k \in \mathbb{R}$  and these undetermined constants shall be selected so that both the numerator and denominator of (5) should not be zero.

### 4. Linear stability Analysis

The Dahlquist's test is applied to the proposed method (3) to determine its stability as below:

$$y' = \lambda h \operatorname{Re}(\lambda) < 0, \quad (6)$$

From this we obtained the difference equations as follows:

$$y_{k+1} = \frac{3(-20y(x) - 16y(x)h\lambda + 8y(x)\lambda h + 4\lambda^2 h^2 y(x) - 4\lambda^2 h^2 y(x) - \lambda^2 h^2 y(x))}{-60y(x) + 12\lambda h y(x) + 24y(x)\lambda h - 3\lambda^2 h^2 y(x) - 6\lambda^2 h^2 y(x) + y(x)\lambda^3 h^3} y_n,$$

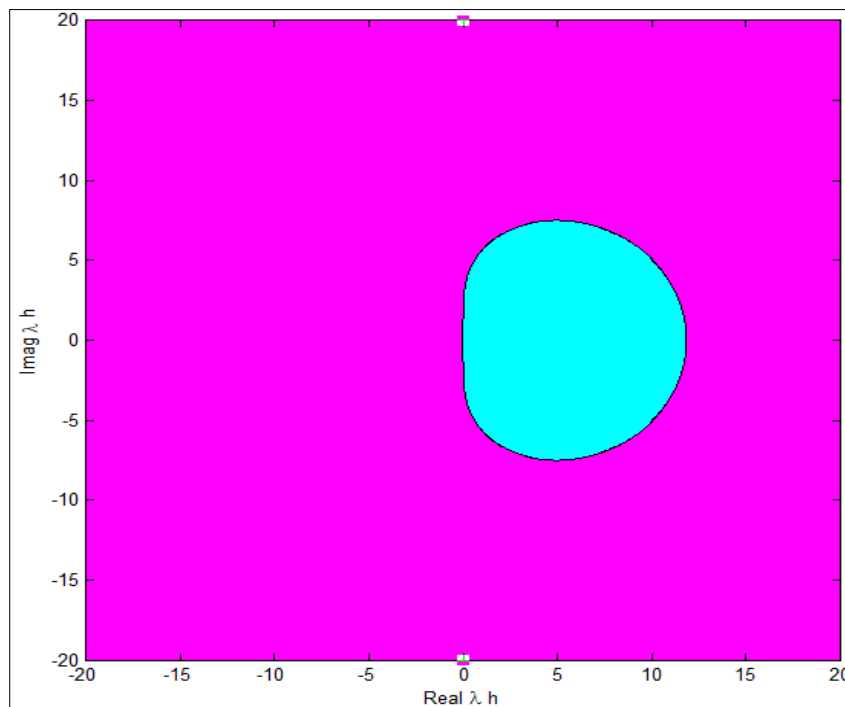
Taking  $y(x)$  common from above equation, we get

$$y_{k+1} = \frac{3(-20-16h\lambda+8\lambda h+4\lambda^2 h^2-4\lambda^2 h^2-\lambda^2 h^2)}{-60+12\lambda h y+24\lambda h-3\lambda^2 h^2-6\lambda^2 h^2+\lambda^3 h^3} y_k.$$

Establishing  $z = \lambda h$  then the stability function defined as follows

$$\phi(z) = \left( \frac{-60-25z-3z^2}{-60+36z-9z^2+z^3} \right). \quad (7)$$

Following figure is showing the stability region inside the closed curve for the proposed method.



**Fig 1:** Showing the stability region inside the closed curve for the proposed method

The figure shows the method is satisfying the A-stability condition and contains the left half complex plane in the region of its absolute convergence. Further, the condition.

$$\lim_{z \rightarrow -\infty} \phi(z) = 0$$

Satisfies which prove that the method is L-stable.

## 5. Numerical test problem

To check the numerical result of proposed method (3), few numerical methods of fifth-order are used for comparison. Particularly, the Taylor's series method and RK5 are chosen for comparison method. Numerical outcome has been considered in term of max error, absolute error and average error. The MATLAB setting of version 8.3.0832 (R2014a) has been utilized for the proposed method. Consider first example from non-linear ordinary differential equation and second example from application problem of population model. Nature of third problem is autonomous and the last example is non-autonomous type of initial value problem.

**Problem1.** Here we have solved a non-linear initial value problem.

$$y'(t) = -2y(t)^2, y(-2) = \frac{1}{5}; -2 \leq t \leq 2$$

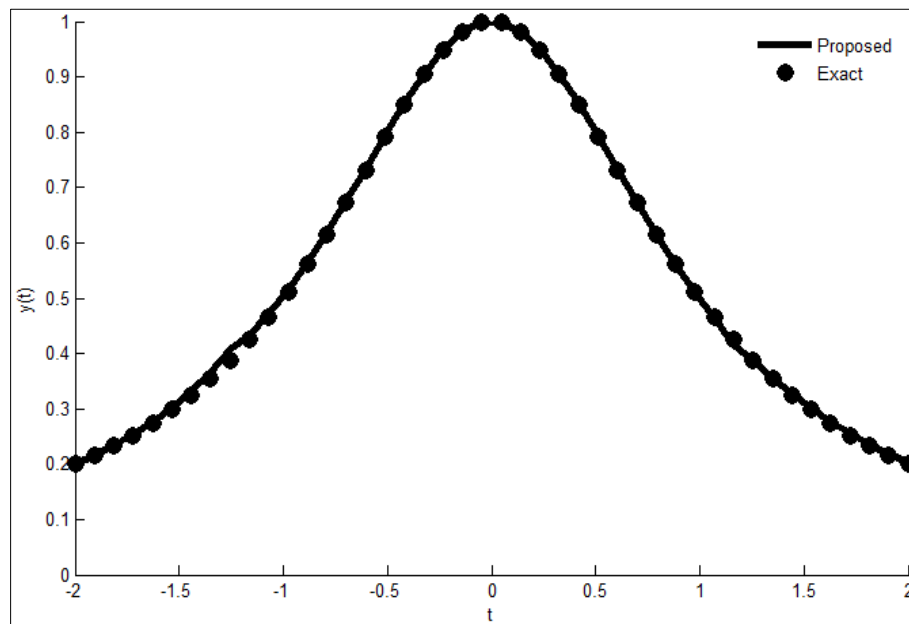


Fig 2

Whose theoretical Solution is given by:  $y(t) = \frac{1}{1+t^2}$

Table 1

| Method\N | No. Of steps |            |            | 512        |
|----------|--------------|------------|------------|------------|
|          | 64           | 128        | 256        |            |
| Taylor   | 6.9359e-11   | 2.1909e-12 | 6.7724e-14 | 7.1054e-15 |
|          | 2.2277e-10   | 9.8167e-12 | 4.3031e-13 | 7.0729e-14 |
|          | 2.0310e-11   | 6.3669e-13 | 2.0278e-14 | 2.4369e-15 |
| Rk5      | 5.0662e-03   | 2.5410e-03 | 1.2725e-03 | 6.3673e-04 |
|          | 1.4280e-02   | 9.9621e-03 | 6.9962e-03 | 4.9300e-03 |
|          | 1.1025e-03   | 5.4717e-04 | 2.7256e-04 | 1.3603e-04 |
| Proposed | 5.5831e-12   | 1.8807e-13 | 1.9540e-14 | 4.5075e-14 |
|          | 3.3347e-11   | 1.5279e-12 | 9.4275e-14 | 3.8454e-13 |
|          | 3.7446e-12   | 1.2108e-13 | 4.9040e-15 | 1.1947e-14 |

First row is showing absolute error at:  $t = 2$ . While maximum absolute error is shown in second row for  $-2 \leq t \leq 2$  and third row is showing average error of each described method for example 1.

Proplem2. Consider the application problem of population model

$y' = ky, k = 1$ , where k is proportionality constant.

Whose theoretical solution is  $y(x) = e^{kx}$ .

Table 2

| Method   | No. Of steps |            |            | 512        |
|----------|--------------|------------|------------|------------|
|          | 64           | 128        | 256        |            |
| Proposed | 3.0409e-07   | 9.4356e-09 | 3.1307e-10 | 8.3844e-12 |
|          | 7.3004e-07   | 3.1298e-08 | 1.4553e-09 | 6.7277e-11 |
|          | 5.0362e-08   | 1.5374e-09 | 5.1067e-11 | 1.9469e-12 |
| Taylor   | 2.8054e-06   | 9.0651e-08 | 2.8807e-09 | 9.0523e-11 |
|          | 6.7351e-06   | 3.0071e-07 | 1.3356e-08 | 5.9004e-10 |
|          | 4.6462e-07   | 1.4770e-08 | 4.6552e-10 | 1.4564e-11 |
| RK5      | 3.0340e+01   | 1.6136e+01 | 8.3290e+00 | 4.2324e+00 |
|          | 7.3433e+01   | 5.3761e+01 | 3.8704e+01 | 2.7619e+01 |
|          | 5.1065e+00   | 2.6514e+00 | 1.3518e+00 | 6.8262e-01 |

First row is showing absolute error at:  $t = 1$ . While maximum absolute error is shown in second row for  $0 \leq t \leq 1$  and third row is showing average error of

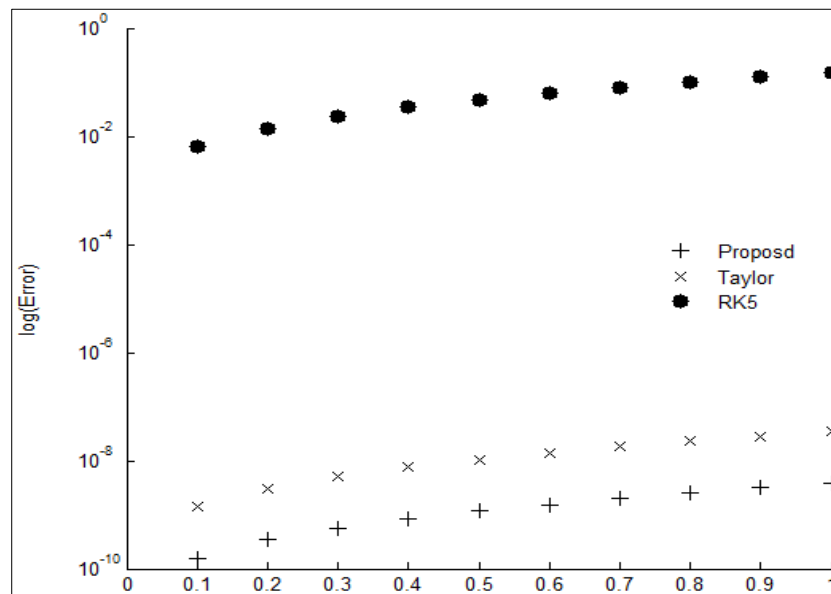


Fig 3: Absolut error for Taylor series method, RK5 method and proposed method for problem2.

Problem3.Consider autonomous problem

$$y' = 1 + y^2 \quad y(0) = 1 \quad 0 \leq t \leq \frac{1}{5}$$

Theoretical solution is  $y(t) = \tan(t + \frac{\pi}{4})$

Table 3

| Method   | No. of steps |            |            | 512        |
|----------|--------------|------------|------------|------------|
|          | 64           | 128        | 256        |            |
| Proposed | 4.5453e-12   | 5.7576e-12 | 1.0339e-11 | 1.2409e-11 |
|          | 4.5453e-12   | 1.4459e-11 | 4.7131e-11 | 5.7299e-11 |
|          | 6.5604e-13   | 6.1489e-13 | 1.3362e-12 | 1.4782e-12 |
| Taylor   | 1.1997e-08   | 3.8966e-10 | 1.2427e-11 | 4.0323e-13 |
|          | 2.3400e-08   | 1.0330e-09 | 4.5665e-11 | 2.0881e-12 |
|          | 1.3904e-09   | 4.3692e-11 | 1.3719e-12 | 4.6267e-14 |
| Rk5      | 1.0076e-01   | 5.2068e-02 | 2.6480e-02 | 1.3354e-02 |
|          | 2.7289e-01   | 1.9475e-01 | 1.3836e-01 | 9.8070e-02 |
|          | 2.2205e-02   | 1.1255e-02 | 5.6667e-03 | 2.8433e-03 |

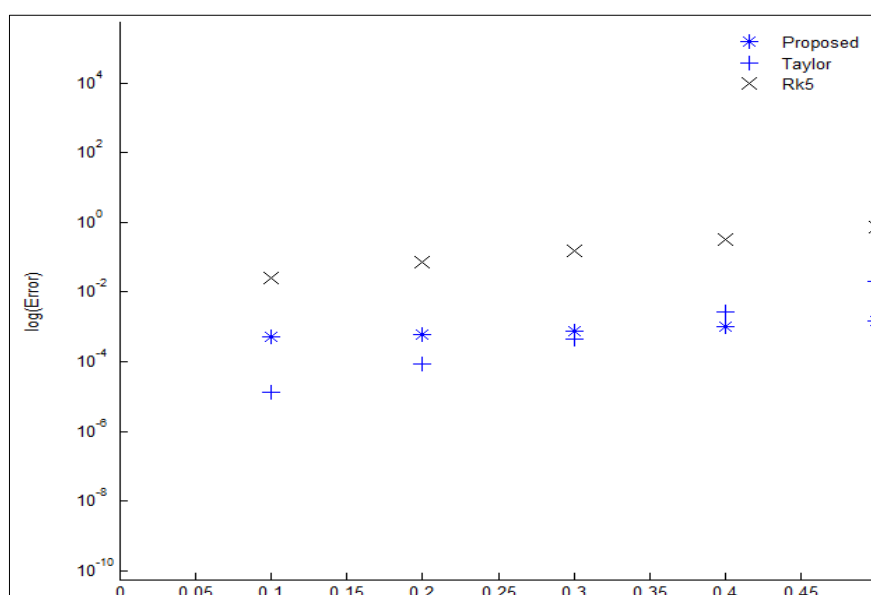


Fig 4: Absolut error for Taylor series method, RK5 method and proposed method for problem3.

First row is showing absolute error at:  $t = 0.5$ . While maximum absolute error is shown in second row for  $0 \leq t \leq 0.5$  and (Third row) is showing average error of each described method for example

#### Problem4. Consider the Non-autonomous of initial value problem

$$y' = t y \quad y(0) = 1 \quad 0 \leq t \leq 1$$

Exact solution is  $y(t) = e^{\frac{t^2}{2}}$

Table 4

| Method\N | No. Of steps |            |            | 512        |
|----------|--------------|------------|------------|------------|
|          | 64           | 128        | 256        |            |
| Taylor   | 6.9359e-11   | 2.1909e-12 | 6.7724e-14 | 7.1054e-15 |
|          | 2.2277e-10   | 9.8167e-12 | 4.3031e-13 | 7.0729e-14 |
|          | 2.0310e-11   | 6.3669e-13 | 2.0278e-14 | 2.4369e-15 |
| Rk5      | 5.0662e-03   | 2.5410e-03 | 1.2725e-03 | 6.3673e-04 |
|          | 1.4280e-02   | 9.9621e-03 | 6.9962e-03 | 4.9300e-03 |
|          | 1.1025e-03   | 5.4717e-04 | 2.7256e-04 | 1.3603e-04 |
| Proposed | 5.5831e-12   | 1.8807e-13 | 1.9540e-14 | 4.5075e-14 |
|          | 3.3347e-11   | 1.5279e-12 | 9.4275e-14 | 3.8454e-13 |
|          | 3.7446e-12   | 1.2108e-13 | 4.9040e-15 | 1.1947e-14 |

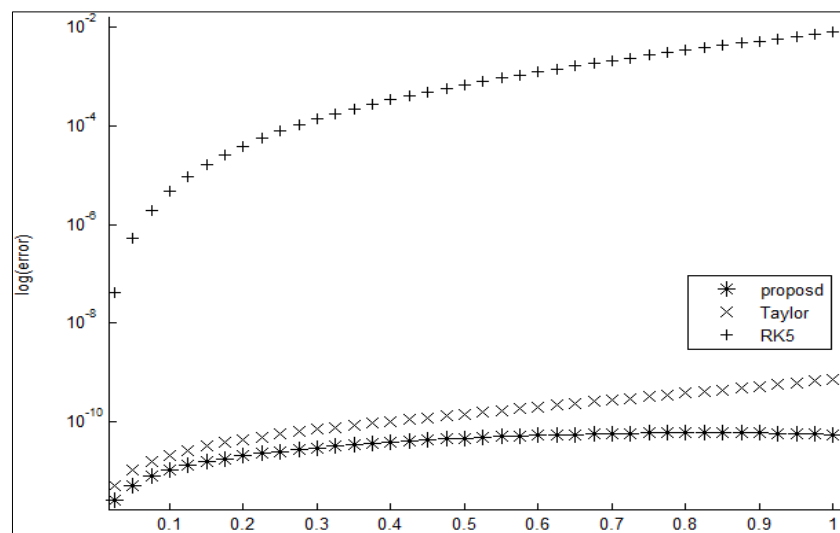


Fig 5: Absolut error for Taylor series method, RK5 method and proposed method for problem4

First row is showing absolute error at:  $t = 1$ . While maximum absolute error is shown in second row for  $0 \leq t \leq 1$  and third row is showing average error of each described method for example 4

#### Result and Discussion

The IVPs of ODEs can be solved easily by using new proposed one-step explicit method (3). Four numerical problems have been solve to check the accuracy level and computational time of the proposed method (3) and compared with two standard numerical methods (Taylor and Rk-5) taken from relevent literature. Approximate result obtained by different step-size are shown using the result analysis table 1-4, and computed absulte error, max error, average error at the final mesh point of the intergration interval. Table 1-4 resolve that small step-size gives better accuracy with less computational error. It may observed from Table 1-4 thatproposed method less error than other methods. Hence the new proposed is superier than Taylor and RK-5 method

#### Conclusion

In this manuscript, a fifth order improved L-stable method has been derived. Also Taylor series method and RK-5 method are compared with the proposed method on initial value problem (IVPs) and proposed method is found more applicable to solve such problems. The local truncation error and stability of proposed method were also investigated. The performance measure of the method is examined on four IVPs. The results and errors obtained via the newly developed scheme shown in Tables 1, 2, 3 and 4 respectively, compared favorably with other existing methods, this proves that the new proposed scheme performs better and is a best choice for solving the IVPs in ODEs. The proposed numerical method is found to be L-stable. Therefore, it is employable for stiff and singular ordinary differential equations.

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