



International Journal of Multidisciplinary Research and Growth Evaluation.

The analysis solution of Non-linear Dirac equation

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Article Info

ISSN (online): 2582-7138

Volume: 03

Issue: 01

January-February 2022

Received: 26-12-2021;

Accepted: 12-01-2022

Page No: 224-228

Abstract

We construct nonlinear extensions of Dirac relativistic electron equation that preserve its other desirable properties such as locality reparability conservation of probability and Poincare invariance we determine the constraint that the nonlinear term must obey and classify the resultant non polynomial nonlinearities in a double expansion in the degree of nonlinearity and number of derivatives. The aims of this paper is to solve the nonlinear Dirac equation using the analysis method. We followed the historical analysis mathematical method and we found the following some results: Possibility of solving the nonlinear Dirac equation for many –bodies system with may bring about more insight into physical system.

Keywords: analysis solution, nonlinear Dirac equation

1. Introduction

The basic of relativistic quantum mechanics was formulated by Paul Dirac in 1928 in away consistent with special relativity [16]. The equation describes the behavior of weakly coupled electrons at high speeds or strongly coupled electrons such as in the case of core electron states in heavy atoms. Among the benefits of this relativistic formulation is the natural emergence of the electron spin and prediction partner to the electron the positron which was discovered experimentally few years later. The physics and mathematics of the Dirac equation is very rich illuminating and provides atheoretical framework for different physical phenomena that are not present in the nonrelativistic regime such as the Klein paradox [22]. In addition Dirac equation emerges in the study of the transport properties in graphene which makes it important for future applications. Exact solution of the Dirac equation with given potential configuration are limited and not trivial compared to the nonrelativistic Schrodinger equation in fact the Dirac Hamiltonian being a matrix in the spinor space allows for more structure in the potential interaction the terminology given to relativistic problem such as the "Dirac –coulomb" "Dirac-oscillation" "Dirac-Morse" "...."etc refers to the Dirac equation that reduces to an effective Schrodinger –like equation with the nominal potential for the large spinor component different approaches were developed to generate exact solution to the Dirac equation such as supersymmetric quantum mechanics and factorization method to only few [29].

2. Analysis

Lemma (2.1)

Let $f/(x^1 \dots x^n), \dots, f^n(x^1 \dots x^n)$ be C^∞ functions of n variables $x^1 \dots x^n$ defined in a neighborhood of $x^1 \dots x^n$ let $d(x^1 \dots x^n) = \det \left(\frac{df^i}{dx^j} \right)$ be the Jacobian determinant and suppose that $J(x^1 \dots x^n) \neq 0$ let $w_0^1 f^1 x^1 \dots x^n, \dots, w_0^n f^n x^1 \dots x^n = f_n(x^1 \dots x^n)$. then there is a neighborhood v of $(x^1, \dots, x^n)$ and there are n C^∞ functions $(w^1, \dots, w^n)$, $g^n(w^1, \dots, w^n)$ satisfying $g^1(w^1, \dots, w^n)$ such that for any $(w^1, \dots, w^n)$. In this neighborhood the equation $(w^1)^1 = f_1(x^1, \dots, x^n), \dots, w^n = f_n(x^1, \dots, x^n)$

Have the unique solution $x^1 = g^1(w^1, \dots, w^n) \dots x, \dots, x^n = g^n(w^1, \dots, w^n)$ such that $(x^1, \dots, x^n) \in v$ [21].

i. Phase Plane Analysis

Example (2.1)

(Pendulum equation for small oscillations) this is the well known equation $\ddot{x} + kx = 0, k > 0$. Solving the equation to obtain the solution explicitly we see that every solution is periodic. The same conclusion may be reached analyzing which turns out to be $(\dot{x}(t))^2 = k(x(t))^2 = 2E$ and E is obtained from the initial conditions the orbits in this case are ellipses (circles if $K=1$) surrounding the origin in the phase plane [1].

3. Plan of Qualitative Analysis

Plan (3.1)

Let us now formulate a plan to qualitatively study a system of two differential equations with two variables and consider several examples we study the system:

$$\begin{cases} \frac{dx}{dt} = f(x, y) \\ \frac{dy}{dt} = g(x, y) \end{cases} \quad (1)$$

Our main aim is to plot the phase portrait of this system and then predict its dynamics. Based on methods which we have developed we will do it in two steps

1. Null- Cline Analysis
2. Jacobian Analysis
3. Null- Cline Analysis

We assume that on the OXY- plane the X –axis is the horizontal axis and the Y- axis is the vertical axis.

1. Draw the $\frac{dx}{dt} = 0$ null- clines from the equation $f(x, y) = 0$ using dashed line and the $\frac{dy}{dt} = 0$ null –clines from the equation $g(x, y) = 0$ using solid line.
2. Choose a point in one of the regions on the X, Y plane and find the vector field for the X-Component denote the X component as dashed $\rightarrow$ if $f(x, y) > 0$ and as dashed $\leftarrow$ if $f(x, y) < 0$.
3. Find the vector field for the Y-component at the same point. Denote the Y –component as solid $\uparrow$ if $g(x, y) > 0$ and as solid $\downarrow$ if $g(x, y) < 0$.
4. Find the vector field in the adjacent regions using the following rule

A. Change the direction of the dashed component of the vector field if to get to the adjacent region you cross the dashed null-cline.

B. Change the direction of the solid component of the vector field if to get to the adjacent region you cross the solid null-cline.

C. Show the direction of the vector field on the null-clines close to the equilibrium [2].

4. The Nonlinear Dirac Equation

To meet the need of simulating self- interacting Dirac fermions [14, 15, 27]. The nonlinear Dirac equation (NLDE) was introduced in 1938 [20] which has the form [11].

$$i\hbar\partial_t\varphi = [-ich\sum_{j=1}^3\alpha_j d_j + mc^2]\varphi + e[v(x, y)I_4 - \sum_{j=1}^3 A_j(t, x)\alpha_j]\varphi + f(\varphi)\varphi \in R^3 \quad (2)$$

The nonlinear Dirac equation (3.1) is similar to the Dirac equation except for the nonlinear term (φ) . The nonlinearity is introduced for self-interaction and in the resulting field equation it is cubic with respect to the wave function which is only significant at extremely high densities at extremely high densities. There have been different cubic nonlinearities generated from different applications [17]. Here we take $f(\varphi) = g_1(\varphi^*\beta\varphi)\beta + g_2|\varphi|^2 I_4 g_1 g_2 \in R$ Two constants and $\varphi^* = \varphi^{-t}$ while f^- denotes the complex conjugate of f . The first term, i.e. $g_1 = 0$ and $g_2 \neq 0$ is generated from and second term i.e. $g_1 = 0$ and $g_2 \neq 0$ is generated from BECS with chiral confinement and/or spin –orbit coupling [10, 19]. A remark is given here that our numerical methods and their error estimates in this thesis can be easily extended to the NLDE with other nonlinearities [26, 27] in fact the fact the NLDE has also been in the Einstein –cartan –Sciama- kibble theory of gravity in order to extend general relativity to matter with intrinsic angular momentum (spin). And recently, the NLDE has been adapted as a mean field model for Bose –Einstein condensates (BECS) and /or cosmology [25]. Moreover the experimental advances in BECS graphene and other 2D materials have also stimulated the research interests on the mathematical analysis and numerical simulations of the Dirac equation and/or the NLDE without /with electromagnetic potentials especially the honeycomb lattice potential [1, 12, 13].

Similar to the process of Dirac equation [5] through a proper non-dimensionalization with the choice of $x_s, t_s = \frac{mx^2}{h}, A_s = \frac{mv^2}{e}$ and $\varphi_s = x_s^{-\frac{3}{2}}$ as the dimensionless length unit time unit potential unit and spinor field unit respectively and dimension reduction we can obtain the dimensionless NLDE in $d=(3,2,1)$ dimensions $i d_1 \varphi = \left[-\frac{i}{\epsilon}\sum_{j=1}^d \alpha_j \partial_j + \frac{1}{\epsilon^2}\beta\right]\varphi + [v(t, x)I_4 \sum_{j=1}^d A_j(t, x)\alpha_j]\varphi + f(\varphi)\varphi \in R^d \quad (3)$

Where ϵ is a dimensionless parameter inversely proportional to the light speed given by

$$0 < \epsilon = \frac{x_s}{t_s} = \frac{v}{c} \leq 1 \quad (4)$$

With $v = \frac{x_s}{t_s}$ the wave speed and

$$f(\varphi) = \lambda_1(\varphi^* \beta \varphi) \beta + \lambda_2 |\varphi|^2 I_4 \varphi \in c^4 \quad (5)$$

Where $\lambda_1 = \frac{g_1}{mv^2 x_s^3} \in R$ and $\lambda_2 = \frac{g_2}{mv^2 x_s^3} \in R$ are two dimensionless constants for the interaction to study the dynamics we give the initial condition $\varphi(t=0, x) = \varphi_0 x \in R^d$ the NLDE (3) is dispersive and time symmetric [29] similar to the Dirac equation after introducing proper total probability density p as well as the current density $J(X, Y) = (J_1(t, x), J_2(t, x))$ we could the conservation law moreover the NLDE (3.2) conserves the total mass. the energy is conserved if the electromagnetic potentials are time independent i.e. $v(t, x) = v(x)$ and $A_j(t, x) = A_j(x)$ for $j = 1, 2, 3$ then $E(t) = \int_{R^d} \left[-\frac{i}{\epsilon} \sum_{j=1}^d \varphi^* \alpha_j \partial_j \varphi + \frac{1}{\epsilon^2} \beta \varphi + v(x) |\varphi|^2 + G(\varphi) - \sum_{j=1}^d A_j(x) \varphi^* \alpha_j \varphi \right] dx$ (6)

Where

$$G(\varphi) = \frac{\lambda_1}{2} (\varphi^* \beta \varphi)^2 + \frac{\lambda_2}{2} |\varphi|^4 \varphi \in c^4 \quad (7)$$

In (3.2) if the external electromagnetic potential are taken to be constants i.e. $v(t, x) \equiv v^\circ$ and $A_j(t, x) \equiv A_j^\circ$ for $j = 1, 2, 3$ then the NLDE (2) admits the plane wave solution as $\varphi(t, x) = B e^{i(k \cdot x - \omega t)}$ where the time frequency ω amplitude vector $B \in R^4$ and spatial wave number $k = (k_1, \dots, k_d) \in R^d$ satisfy.

$$\omega = \left[\sum_{j=1}^d \left(\frac{k_j}{\epsilon} - A_j^\circ \right) \alpha_j + \frac{1}{\epsilon^2} \beta + v^\circ I_4 + \lambda_1 (B^* \beta B) \beta + \lambda_2 |B|^2 I_4 \right] B \quad (8)$$

Which immediately gives the dispersion relation of the NLDE (2) as

$$\omega = \omega(k, B) = V^\circ + \lambda_2 |B|^2 \pm \frac{1}{\epsilon^2} \sqrt{[1 + \epsilon^2 \lambda_1 (B^* \beta B)] + \epsilon^2 |k - \epsilon A^\circ|^2} k \in R^d \quad (9)$$

Again similar to the Dirac equation [5] for one dimension (1D) and two dimension (2D) the NLDE (3.2) can be simplified to the following one

$$i d_t \Phi = \left[-\frac{i}{\epsilon} \sum_{j=1}^d \sigma_j d_j + \frac{1}{\epsilon^2} \sigma_3 \right] \Phi + [v(t, x) I_2 - \sum_{j=1}^d A_j(t, x) \sigma_j] \Phi + f(\Phi) \Phi x \in R^d \quad (10)$$

Where

$$f(\Phi) = \lambda_1 (\Phi^* \sigma_3 \Phi) \sigma_3 + \lambda_2 |\Phi|^2 I_2 \Phi \in c^2 \quad (11)$$

With λ_1 and λ_2 both real numbers in (3.9) the two-component wave functions Φ defined as $\Phi = \Phi(t, x) = (\Phi_1(t, x), \Phi_2(t, x))^t \in c^2$ the initial condition for dynamics is given as [19].

$$\Phi(t=0, x) = \Phi_0(x) x \in R^d \quad (12)$$

The NLDE (8) has similar properties to its four-component version (2) it is dispersive and time symmetric satisfies the conservation law {7} conserves total mass and also conserves energy

$$E(t) = \int_{R^d} \left(-\frac{i}{\epsilon} \sum_{j=1}^d \Phi^* \sigma_j d_j \Phi + \frac{1}{\epsilon^2} \Phi^* \sigma_3 \Phi + v(x) |\Phi|^2 - \sum_{j=1}^d A_j(x) \Phi^* \sigma_j \Phi + G(\Phi) \right) dx \quad (13)$$

Where

$$G(\Phi) = \frac{\lambda_1}{2} (\Phi^* \sigma_3 \Phi) + \frac{\lambda_2}{2} |\Phi|^4 \Phi \in c^2 \quad (14)$$

If the electromagnetic potentials time-independent.

Under constant external electromagnetic potentials i.e. $v(t, x) \equiv v^\circ$ and $A_j(t, x) \equiv A_j^\circ$ for $j = 1, 2$ the NLDE (8) admits the plane wave solution as $\Phi(t, x) = B e^{i(k \cdot x - \omega t)}$ with the time frequency ω amplitude vector $B \in R^2$ and spatial wave number $k = (k_1, \dots, k_d)^t \in R^d$ satisfy.

$$wB = \left[\sum_{j=1}^d \left(\frac{ki}{\varepsilon} - A_j^\circ \right) \sigma_j + \frac{1}{\varepsilon^2} \sigma_3 + v^\circ I_2 + \lambda_1 (B^* \sigma_3 B) \sigma_3 + \lambda_2 |B|^2 I_2 \right] B \quad (15)$$

With implies dispersion relation of the NLDE (3.9) directly as

$$w = w(k, B) = v^\circ + \lambda_2 |B|^2 \pm \frac{1}{\varepsilon^2} \sqrt{(1 + \varepsilon^2 \lambda_1 (B^* \sigma_3 B))^2 + \varepsilon^2 |k - \varepsilon A^\circ|^2} k \in \mathbb{R}^d \quad (16)$$

The NLDE (2) has different regimes with different choices of the dimensionless parameter ε . When $\varepsilon = 1$ which corresponds to the classical regime extensive analytical and numerical result have been obtained [3, 4, 9].

5. Solution of Nonlinear Dirac Equation

Theorem (5.1)

Let $(W), (v)$ and $(f_5) - f_2, (f_3'), f_4'$ be satisfied. then (D) v has at least one nontrivial $u \in \cap_{t \geq 2} W^{i,t}(R^3, C^4)$ if in addition f is even and (f_5) hold then $(D)_v$ has infinitely many geometrically distance solution $U \in \cap_{t \geq 2} W^{i,t}(R^4, C^4)$. Here are examples where the assumption apply.

Example (5.2)

- $f(x, u) = \frac{1}{2} b(x) |u|^2 \left(1 - \frac{1}{\ln(e+|u|)} \right)$.
- $f(x, u) = b(x) \psi \left(\frac{1}{2} \right) |u|^2$ where $\psi = [0, \infty) \rightarrow [0, \infty)$ is of class C^2 with $\varphi(0) = \varphi'(0) = 0$ and $\varphi'(s) \rightarrow 1$ as $s \rightarrow \infty$ and $\varphi''(s) \geq 0$.
- $f_u(x, u) = f(x, |u|)u$ where $f(x, s)$ is even in s , $f(x, s) \rightarrow 0$ as $s \rightarrow 0$ uniformly in x , $f(x, s)$ is nondecreasing for $s \in [0, \infty)$ and $f(x, s) \rightarrow b(x)$ as $s \rightarrow \infty$.

6. Variational Setting

We denote by $|\cdot|_p$ the usual L^p norm for $p \in [1, \infty]$. For $v \in L^2_{100}(R^3, R)$ the operator $A = -i \sum_{k=1}^3 a_k \partial_k + (v(x) + a)\beta$ is a self-adjoint operator in $l^2 = l^2(R^3, C^4)$ it is unbounded from above and from below in order to investigate the spectrum of A we consider

$$A^2 = \Delta + (V + a)^2 + i \sum_{k=1}^3 \beta a_k d_k v$$

Let $\sigma(s), \sigma_d(s), \sigma_e(s)$ and $\sigma_c(s)$ denote respectively the spectrum, the essential spectrum and the continuous spectrum of a self-adjoint operator S on L^2 .

Result

We gave several examples of such equations different in structure from those studied previously in the literature and discussed their properties. We also demonstrated that our equation were not gauge equivalent to the linear Dirac equation. Solution to our nonlinear equation similar to what has been done for simpler polynomial type nonlinear Dirac equation.

Conclusion

We found the possibility of solving the nonlinear Dirac equation for many n -bodies system using analysis solution.

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