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A general class of generating functions through group theoretic approach and its applications

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Abstract

In the present paper, we introduce a general class of generating functions involving the product of modified Bessel polynomials $Y_n^{\alpha+n}(.)$ and the confluent hypergeometric function ${}_1F_1(.)$ and then, obtain its some more general class of generating functions by group-theoretic approach and discuss their applications.

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1. Introduction

Krall and Frink [1] introduced generalized Bessel polynomials defined by

$$Y_n^{\alpha}(x) = {}_2F_0[-n, n+\alpha-1; -; -x/\beta] \qquad (1.1)$$

Further Mukherjee and Chongdar [4] have considered and studied the modified Bessel polynomials defined by

$$Y_n^{\alpha+n}(x) = {}_2F_0[-n, 2n+\alpha-1; -; -x/\beta] \qquad (1.2)$$

The function ${}_2F_0(.)$ can be replaced by many special functions such as

The Leguerre polynomials or the parabolic cylinder functions etcetera

Srivastava and Manocha [3] defined and studied various bilinear, bilateral and multilinear functions. Chatterjee and Chakraborty [5] introduced and studied some quasi-bilinear and quasi-bilateral generating functions. Mukherjee [8] defined and studied an extension of bilateral generating functions of certain special functions. Rama Kameswari and Bhagavan [14] introduced and studied group theoretic origins of certain generating functions of Legendre Polynomials. Further Bhandari [15] defined and studied general class of generating functions and its applications.

Motivated by above work, in the present paper, we introduce the following new general class of generating functions:

$$G(x,u,w) = \sum_{n=0}^{\infty} A_n w^n Y_n^{\alpha+n}(x)\, {}_1F_1\left[\begin{matrix}-n; \\ 1+n;\end{matrix}\ u\right] \qquad (1.3)$$

Where, an is any arbitrary sequence independent of x, u and w. Since in (1.3) as setting various values of An, we may find several results on generating functions involving different special functions.
Further, making an appeal to the group-theoretic techniques, here in the present paper, we evaluate some more general class of generating functions and finally discuss their applications.

2. Group-Theoretic Operators.

In our investigations, we use the following group-theoretic-operators and their actions:
The operator H1 due to Kar [7] is given by

$$H_1 = x^2yz^{-2}\frac{\partial}{\partial x} + 2xy^2z^{-2}\frac{\partial}{\partial y} + xyz^{-1}\frac{\partial}{\partial z} + (\beta - x)yz^{-2} \tag{2.1}$$

Such that

$$H_1[Y_n^{\alpha+n}(x)y^nz^{\alpha}] = \beta Y_{n+1}^{\alpha+n-1}(x)y^{n+1}z^{\alpha'-2} \tag{2.2}$$

The operator H2 due to Miller Jr. [2] is given by

$$H_2 = v\frac{\partial}{\partial t} + vut^{-1}\frac{\partial}{\partial u} - vut^{-1} \tag{2.3}$$

Such that

$$H_2\left[{}_1F_1\left[\begin{matrix}-n; \\ m+1;\end{matrix} u\right] v^nt^m\right] = m\,{}_1F_1\left[\begin{matrix}-n-1; \\ m;\end{matrix} u\right] v^{n+1}t^{m-1} \tag{2.4}$$

The actions of H1 and H2 on f are obtained as follows:

$$\exp[wH_1]f(x,y,z) = \left(1 - \frac{wxy}{z^2}\right) exp\left[\frac{wxy}{z^2}\right] f\left(\frac{x}{1-\frac{wxy}{z^2}}, \frac{y}{\left(1-\frac{wxy}{z^2}\right)^2}, \frac{z}{1-\frac{wxy}{z^2}}\right) \tag{2.5}$$

And

$$\exp[wH_2]f(v,t,u) = exp\left[\frac{-uvw}{t}\right] f\left(v, t+wv, u\left(1+\frac{wv}{t}\right)\right) \tag{2.6}$$

3. Some more general class of generating function

In this section, making an use of the general class of generating function (1.3) and group-theoratic operators H1 and H2 with their actions given in the Section 2, we obtain some more general class of generating functions through following theorem:
Theorem. If there exists a general class of generating functions involving the product of modified Bessel polynomials and the confluent hypergeometric functions given by (1.3), then following more general class of generating functions hold:

$$(1+w)^m(1-wx)^{1-\alpha}\exp[w(\beta-u)]G\left[\frac{x}{1-wx}, u(1+w), \frac{wyt}{(1-wx)^2}\right]$$
$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} A_n \frac{(-1)^s\Gamma(m+1)}{r!\,s!\,\Gamma(m-s+1)} Y_{n+r}^{\alpha+n-r}(x)\,{}_1F_1\left[\begin{matrix}-n-s; \\ m-s+1;\end{matrix} u\right] (wyt)^n(\beta w)^r(-w)^s \tag{3.1}$$

$$(1+w)^m(1+wx)^{1-\alpha}\exp[-w(\beta+u)]G\left[\frac{x}{1+wx}, u(1+w), \frac{wyt}{(1+wx)^2}\right]$$
$$= \sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} A_n \frac{(-1)^s\Gamma(m+1)}{r!\,s!\,\Gamma(m-s+1)} Y_{n+r}^{\alpha+n-r}(x)\,{}_1F_1\left[\begin{matrix}-n-s; \\ m-s+1;\end{matrix} u\right] (wyt)^n(-\beta w)^r(-w)^s \tag{3.2}$$

or equivalently

$$(1+w)^m(1-wx)^{1-\alpha}\exp[w(\beta-u)]G\left[\frac{x}{1-wx},u(1+w),\frac{wyt}{(1-wx)^2}\right],$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{r}A_n\frac{(-r)_n(-1)^s\Gamma(m+1)}{r!\,s!\,\Gamma(m-s+1)}Y_{n+r}^{\alpha+2n-r}(x)\,{}_1F_1\left[\begin{matrix}-n-s;\\ m-s+1;\end{matrix}u\right]\;(-yt)^n(\beta)^{r-n}(-w)^s w^r \tag{3.3}$$

and

$$(1+w)^m(1+wx)^{1-\alpha}\exp[-w(\beta+u)]G\left[\frac{x}{1+wx},u(1+w),\frac{wyt}{(1+wx)^2}\right]$$

$$=\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}\sum_{n=0}^{r}A_n\frac{(-r)_n(-m)_s}{r!\,s!}Y_r^{\alpha+2n-r}(x)\,{}_1F_1\left[\begin{matrix}-n-s;\\ m-s+1;\end{matrix}u\right]\;(yt)^n(\beta)^{r-n}(-w)^s(-w)^r. \tag{3.4}$$

Proof: In the general class of generating function (1.3), replacing w by wyv and then multiplying by $z\alpha$ tm both sides, we get

$$G(x,u,wyv)z^\alpha t^m=\sum_{n=0}^{\infty}A_n w^n y_n^{\alpha+n}(x)y^n z^\alpha\,{}_1F_1\left[\begin{matrix}-n;\\ m+1;\end{matrix}u\right]v^n t^m. \tag{3.5}$$

Now, making an appeal to (2.2) and (2.4), from (3.5), we derive

$$\exp[wH_1]\{Y_n^{\alpha+n}(x)y^n z^\alpha\}=\sum_{n=0}^{\infty}\frac{(w)^r}{r!}\beta^r y_{n+r}^{\alpha+n-r}(x)y^{n+r}z^{\alpha-2r} \tag{3.6}$$

and

$$\exp[wH_2]\left\{{}_1F_1\left[\begin{matrix}-n;\\ m+1;\end{matrix}u\right]v^n t^m\right\}$$

$$=\sum_{n=0}^{\infty}\frac{(-w)^s}{s!}\frac{(-1)^s\Gamma(m+1)}{\Gamma(m-s+1)}\,{}_1F_1\left[\begin{matrix}-n-s;\\ m-s+1;\end{matrix}u\right]\;v^{n+s}t^{m-s}. \tag{3.7}$$

Now, operating both sides of (3.5) by the operators exp[$wH1$]exp[$wH2$] and then, making an appeal to the relations (2.5) and (2.6) in the left hand side and (3.5) and (3.6) in the right hand side, we evaluate

$$z^\alpha(t+wv)^m(1-wxy/z^2)^{1-\alpha}\exp[w(\beta y/z^2-vu/t)]$$

$$G\left[\frac{x}{1-wxy/z^2},u(1+wv/t),\frac{wyt}{(1-wxy/z^2)^2}\right]$$

$$=\sum_{n=0}^{\infty}\sum_{s=0}^{\infty}\sum_{r=0}^{\infty}A_n\frac{(w)^{n+r+s}\Gamma(m+1)}{r!\,s!\,\Gamma(m-s+1)}Y_{n+r}^{\alpha+n-r}(x)y^{n+r}z^{\alpha-2r}\,{}_1F_1\left[\begin{matrix}-n-s;\\ m-s+1;\end{matrix}u\right]\;v^{n+s}t^{m-s} \tag{3.8}$$

Now, setting $y/z2$ = 1 and v=t in (3.8), we prove (3.1).

Again, setting $y/z2$ = −1 and v=t in (3.8), we prove (3.2).

Finally, replacing r by r-n and than applying series rearrangement techniques in (3.1) and (3.2), we obtain (3.3) and (3.4) respectively.

4. Special Cases: Applications and Deductions.

For ma positive integer (3.1), (3.2), (3.3) and (3.4) reduce respectively to

$$(1+w)^m(1-wx)^{1-\alpha}\exp[w(\beta-u)]G\left[\frac{x}{1-wx},u(1+w),\frac{wyt}{(1-wx)^2}\right]$$
$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{m}A_n\frac{(-m)_s}{r!\,s!}Y_{n+r}^{\alpha+n-r}(x)\,{}_1F_1\left[\begin{matrix}-n-s;\\ m-s+1;\end{matrix}u\right](wyt)^n(\beta w)^r(-w)^s \tag{4.1}$$

$$(1+w)^m(1+wx)^{1-\alpha}\exp[-w(\beta+u)]G\left[\frac{x}{1+wx},u(1+w),\frac{wyt}{(1+wx)^2}\right]$$
$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{m}A_n\frac{(-m)_s}{r!\,s!}Y_{n+r}^{\alpha+n-r}(x)\,{}_1F_1\left[\begin{matrix}-n-s;\\ m-s+1;\end{matrix}u\right](wyt)^n(-\beta w)^r(-w)^s \tag{4.2}$$

$$(1+w)^m(1-wx)^{1-\alpha}\exp[w(\beta-u)]G\left[\frac{x}{1-wx},u(1+w),\frac{wyt}{(1-wx)^2}\right]$$
$$=\sum_{r=0}^{\infty}\sum_{n=0}^{r}\sum_{s=0}^{m}A_n\frac{(-r)_s(-m)_s}{r!\,s!}Y_{r}^{\alpha+m-r}(x)\,{}_1F_1\left[\begin{matrix}-n-s;\\ m-s+1;\end{matrix}u\right](-yt)^n(\beta)^{r-n}(w)^r \tag{4.3}$$

$$(1+w)^m(1+wx)^{1-\alpha}\exp[-w(\beta+u)]G\left[\frac{x}{1+wx},u(1+w),\frac{wyt}{(1+wx)^2}\right]$$
$$=\sum_{r=0}^{\infty}\sum_{n=0}^{r}\sum_{s=0}^{m}A_n\frac{(-r)_n(-m)_s}{r!\,s!}Y_{r}^{\alpha+2n-s}(x)\,{}_1F_1\left[\begin{matrix}-n-s;\\ m-s+1;\end{matrix}u\right](yt)^n(\beta)^{r-n}(-w)(-w)^r \tag{4.4}$$

From (4.1), we further derive

$$(1+w)^m(1-wx)^{1-\alpha}\exp[w(\beta-u)]G\left[\frac{x}{1-wx},u(1+w),\frac{wyt}{(1-wx)^2}\right]$$
$$=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{m}A_{n-r}\frac{(-m)_s}{r!\,s!}Y^{\alpha+n-2r}(x)\,{}_1F_1\left[\begin{matrix}-(n+s-r);\\ m-s+1;\end{matrix}u\right](yt)^{n-r}w^n(-w)^s(\beta w)^r \tag{4.5}$$

While from (4.2), we obtain

$$(1+w)^m(1+wx)^{1-\alpha}\exp[-w(\beta+u)]G\left[\frac{x}{1+wx},u(1+w),\frac{wyt}{(1+wx)^2}\right]$$
$$=\sum_{n=0}^{\infty}\sum_{r=0}^{n}\sum_{s=0}^{m}A_{n-r}\frac{(-m)_s}{r!\,s!}Y_n^{\alpha+n-2r}(x)\,{}_1F_1\left[\begin{matrix}-(n+s-r);\\ m-s+1;\end{matrix}u\right](yt)^{n-r}w^n(-\beta)^r(-w)^s \tag{4.6}$$

Further setting β = u and t=1 in (3.1), we derive

$$(1+w)^m(1-wx)^{1-\alpha}G\left[\frac{x}{1-wx},u(1+w),\frac{wy}{(1-wx)^2}\right]$$
$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}A_n\frac{m!\,n!}{(m+n)!}\frac{(1+n)_s}{r!\,s!}Y_{n+r}^{\alpha+n-r}(x)L_{n+s}^{(m-s)}(u)(wy)^n(uw)^r(w)^s, \tag{4.7}$$

Where are $(m)(u)$ Leguerre polynomials.
For $\beta=-u$, $t=1$, (3.2) gives

$$(1+w)^m(1+wx)^{1-\alpha}G\left[\frac{x}{1+wx},u(1+w),\frac{wy}{(1+wx)^2}\right]$$
$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty}A_n\frac{m!\,n!}{(m+n)!}\frac{(1+n)_s}{r!\,s!}Y_{n+r}^{\alpha+n-r}(x)L_{n+s}^{(m-s)}(u)(wy)^n(uw)^r(w)^s. \tag{4.8}$$

Other similarly results can be obtained form (3.3) and (3.4) in similar manner.
If m is positive integer than (4.7) and (4.8) give

$$(1+w)^m(1-wx)^{1-\alpha}G\left[\frac{x}{1-wx},u(1+w),\frac{wy}{(1-wx)^2}\right]$$
$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{m}A_n\frac{m!\,n!}{(m+n)!}\frac{(1+n)_s}{r!\,s!}Y_{n+r}^{\alpha+n-r}(x)L_{n+s}^{(m-s)}(u)(wy)^n(uw)^r(w)^s. \tag{4.9}$$

and

$$(1+w)^m(1+wx)^{1-\alpha}G\left[\frac{x}{1+wx},u(1+w),\frac{wy}{(1+wx)^2}\right]$$
$$=\sum_{n=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{m}A_n\frac{m!\,n!}{(m+n)!}\frac{(1+n)_s}{r!\,s!}Y_{n+r}^{\alpha+n-r}(x)L_{n+s}^{(m-s)}(u)(wy)^n(uw)^r(w)^s. \tag{4.10}$$

Respectively.

Further setting m=0 and t=1 in (3.1), we derive a generating relation

$$(1-wx)^{1-\alpha}\exp(\beta w)G\left[\frac{x}{1-wx},\frac{wy}{(1-wx)^2}\right]$$
$$=\sum_{r=0}^{\infty}\sum_{n=0}^{r}A_n\frac{w^r}{(r-n)!}\beta^{r-n}Y_r^{\alpha+2n-r}(x)y^n, \tag{4.11}$$

which is similar result due to Mukherjee and Chongdar [4],

while for m=0 and t=1, from (3.2), we obtain a generating relation:

$$(1+wx)^{1-\alpha}\exp(-\beta w)G\left[\frac{x}{1+wx},u(1+w),\frac{wy}{(1+wx)^2}\right]$$
$$=\sum_{r=0}^{\infty}\sum_{n=0}^{r}A_n\frac{w^r}{(r-n)!}(-\beta)^{r-n}y_r^{\alpha+2n-r}(x)y^n, \tag{4.12}$$

From (4.7) and (4.8), we further derive a relation

$$(1-wx)^{1-\alpha}G\left[\frac{x}{1-wx},u(1+w),\frac{wy}{(1-wx)^2}\right]$$
$$=(1+wx)^{1-\alpha}G\left[\frac{x}{1+wx},u(1+w),\frac{wy}{(1+wx)^2}\right]. \tag{4.13}$$

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