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Comparison between Newtonian fluid flow, Stokes flow and Stokes-Oseen flow

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Abstract
Viscosity is an intrinsic property of fluids. It originates in the thermal interactions of molecular collisions. There are two types of it: kinematic viscosity and dynamic viscosity. In this work, we try to model the flow of viscous fluid in three cases: Newtonian fluid flow, Stokes flow and the Stokes-Oseen flow.

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1. Introduction
To represent a fluid flow, there are two modeling approaches:
1. In a Lagrangian approach, the movement of the fluid, over time, is highlighted. In particular, trajectories and emission lines are measured or calculated.
2. In an Eulerian approach, the velocity field is highlighted, at a given instant, over the entire spatial domain studied. In particular, the velocity vectors are represented at numerous points or else current lines.

In this work, we present the Eulerian equations of the flow of a fluid.

2 Newtonian fluid flow
These fluids are characterized by particular rheological and thermal behaviors. Indeed, the shear rate tensor (describing the tangential shear stresses) and the current heat vector are expressed linearly respectively as a function of the strain rate tensor (characterizing the strain rate) and the temperature gradient vector. The local equations describing the dynamics of Newtonian fluids have the form of the Navier-Stockes equation. But there is a problem: the number of equations is less than the number of variables.
The modeling of flow phenomena is governed essentially by three fundamental principles in physics
- The mass is preserved
- The variation in momentum (mass × velocity) is equal to the sum of the applied forces;
- The total energy is conserved: this is the first principle of thermodynamics.

They essentially contain three variables: the velocity v, the pressure p and the temperature T. The formulation of the so-called Eulerian global equations is obtained by integrating the elementary and local equations of mass, energy and momentum. In this model, two other energy equations are added. The equations are:

Conservation of mass

$$\frac{d\rho}{dt} + \nabla.(\rho v) = 0 \qquad (1)$$

Conservation of momentum

$$\rho\left[\frac{dv}{dt} + v\nabla.v\right] = \rho f - \nabla p + \nabla.\tau \qquad (2)$$

Energy conservation

$$\rho\left[\frac{d}{dt}\left(\frac{v^2}{2}+e\right) + v.\nabla\left(\frac{v^2}{2}+e\right)\right] = \rho f.v - \nabla.(\sigma.v) + \nabla.q + \rho r \qquad (3)$$

Kinetic energy balance

$$\rho\left[\frac{de_c}{dt} + v.\nabla e_c\right] = \rho f.v - \nabla.(\sigma.v) + \tau : (\nabla v) \qquad (4)$$

with $e_c = \frac{1}{2}\rho v^2$.

Internal energy balance

$$\rho\left[\frac{de}{dt} + v.\nabla e\right] = -\nabla.q + \rho r + \tau : (\nabla v) \qquad (5)$$

These equations form an ill posed mathematical problem because the variables are more numerous than the equations. Indeed, the equations are deduced from each other. For example, the equation of the kinetic energy 4 is deduced from that of the momentum 2. In addition, the internal energy balance 5 is none other than the difference between 3 and 4. In conclusion, the system contains only three independent equations to choose from, for example, the equations 1, 2 and 5. The variables of this new system formed by three equations are:

- velocity v,
- pressure p,
- temperature T
- density ρ
- mass density of the forces at a distance f
- mass density of the radiated heat power r
- tensor of shear rates τ
- current heat vector q
- mass density of the internal energy e

Solving such a problem can be solved by several methods. The first is to use the theory of inverse problems. The second consists in using the constitutive laws of Newtonian fluids. In this case, it is necessary to supplement the system of equations with six more equations. Thus, the system of equations will contain only three variables: the velocity v, the pressure p and the temperature T. For more details concerning the constitutive laws of a Newtonian fluid, the reader may refer to [11].
The closed system with three variables deduced in the same reference contains two hydrodynamic equations and a thermal equation. The first two are called the Navier-Stokes equations. The last one is called the heat equation.
They are:

Stokes equation

$$\nabla.v = 0 \qquad (6)$$

Navier-Stokes equation

$$\frac{\partial v}{\partial t} + v.\nabla v = -\frac{1}{\rho}\nabla p + v\nabla\tau + g \qquad (7)$$

Heat Equation

$$\rho C_v \left[\frac{\partial T}{\partial t} + v.\nabla T\right] = \Delta T + \rho r + \mu \left[\nabla v + (\nabla v)^t\right] : (\nabla v) \quad (8)$$

The last term in this equation shows the calorific power produced within the flow by viscous friction between the fluid particles. The two equations of the hydrodynamic system form a closed system, that is to say solvable. With adequate edge conditions, they form a well-posed problem. The velocity v(x,t) and pressure p(x,t) fields are the only unknown ones. The Navier-Stokes system makes it possible to model the flows of incompressible Newtonian fluids in any environment and at any scale, macroscopic or microscopic. However, these equations are adaptable to the situations studied. For example, sometimes the convective term v.∇v is taken into consideration, other times it is completely neglected.

The heat equation is also independent of the fluid dynamics equations. It is also solvable in some cases with precise edge conditions. For more details concerning the demonstration of Eulerian equations, the reader can refer to the course [12].

3. Stokes flow

A Stokes flow (or creeping flow) characterizes a viscous fluid that flows slowly in a narrow place or around a small object. Viscous effects then dominate over inertial effects. We sometimes speak of Stokes fluid as opposed to the perfect fluid. It is modeled by the Stokes equation. In it, inertial terms are absent. This affects the Reynolds number. Indeed, as it measures the relative weight of the viscous and inertial terms in the Navier-Stokes equation. In a Stokes flow, the Reynolds number is very low it is much smaller than 1.

The Stokes equation makes it possible in particular to describe the speed of very small particles in liquids or even gases.

Example 1: The case of ice crystals is clarified in the upper atmosphere or common fog, by Stokes modeling.

The Stokes equation also makes it possible to describe the liquid flows in the microfluidic devices as well as the Couette flow and that of Poiseuille. Unlike the Navier-Stokes equation, the Stokes equation is linear. As explained above, the inertial, non-linear term is negligible. The flow velocities solutions of this equation therefore have very particular properties: uniqueness, additivity and reversibility.

The Stokes equation can be solved analytically to calculate the force exerted on a sphere of radius r. The Stokes equation has been solved for bodies other than the sphere. Thus Oberbeck obtained as early as 1876 the exact value of the drag of the elongated or flattened ellipsoid of revolution.

Example 2: In the book [5], a transition from the macroscopic scale to the microscopic scale influences the physical laws taken into consideration. For example, the author describes small-scale blood flow as Stokes flow. Indeed, by changing scale, several phenomena seem to be quasi-stationary, or even non-existent as is the case of convection. On a macroscopic scale, this phenomenon has its importance in the transfer of energy and the improvement of the separation of the particles from the fluid.

The mathematical solutions of the Stokes equation, in a particular case or in certain regions of the mathematical domain of solutions, can be physically false. This is called the "Stokes paradox". Indeed, the physical conditions emitted to transform the Navier-Stokes equation into the Stokes equation are not necessarily realized in the entire solution domain. Therefore, solutions have potentially aberrant behaviors within certain limits. As the example of the condition "at infinity", since the inertial term prevails over the viscous term, in most cases. This paradox is solved thanks to the equations of Carl Wilhelm Oseen in 1910.

4. Stokes-Oseen Flow

In fluid dynamics, the Stokes-Oseen equations describe the flow of an incompressible viscous fluid for a low Reynolds number. The Oseen equations models an object moving at a low speed U, in an immobile fluid, the flow is described in a frame of reference linked to the object by the following equations:

$$\begin{cases} -\rho U.\nabla u & = & -\nabla p + \mu \nabla^2 u \\ \nabla.u & = & 0 \end{cases} \quad (9)$$

With

- u(r) is the velocity of the fluid in the reference system
- p(r) is the pressure of the fluid
- ρ is the density
- µ is the dynamic viscosity of the fluid
- ∇ is the gradient operator
- ∇2 is the Laplacian operator

The Oseen problem solves the Stokes paradox, as explained in the figure 1.

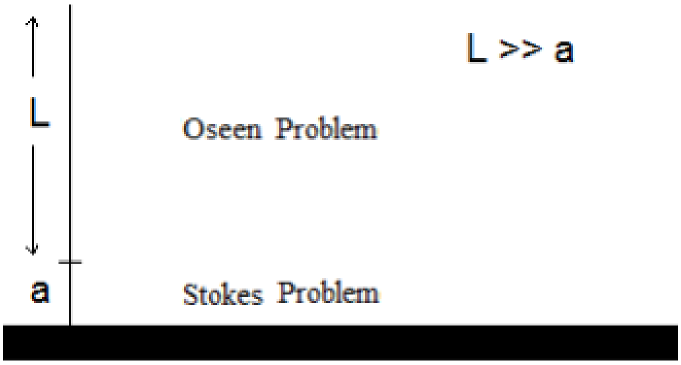

Fig 1: The Stokes paradox: the areas of validity of the Stokes and Oseen problems

5. Conclusion

In this article, the author studies three particular cases of a flow movement. The objective is to model the flow of viscous fluid in microsopic scale.

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