Nonlinear PDE in variable exponent Sobolev spaces

Yosra Annabi
Free Researcher, Street Cheikh Mohamed Hedi Belhadej, Romana, Tunisia

* Corresponding Author: **Yosra Annabi**

Article Info

ISSN (online): 2582-7138
Volume: 05
Issue: 05
September-October 2024
Received: 12-07-2024
Accepted: 14-08-2024
Page No: 252-258

Abstract
In this article, the author introduces the Lebesgue and Sobolev functional spaces with variable exponents. For this, she proposes a direct problem to be solved in these spaces. In addition, she presents a comparison of convergence theorems in spaces with fixed exponents and those with variable exponents.

DOI: https://doi.org/10.54660/.IJMRGE.2024.5.5.252-258

Keywords: EDP, Nonlinear, Functional spaces, Lebesgue, Dirichlet problem, p-Laplacian, Sobolev spaces

1. Introduction
In this article we consider the following problem:
Show the existence and uniqueness of a weak solution of the following nonlinear Dirichlet problem:

$$\textbf{Syst QES:} \quad \begin{array}{l} -\operatorname{div}(\sigma(x,u(x),Du(x))) = v(x) \\ u = 0 \text{ on } \partial\Omega \end{array} \qquad (1)$$

With appropriate assumptions about σ, v, f et g.
Example 1 Example: p-Laplacien. The QES system can be generalized to the p-Laplace operator Δp:

$$\sigma(x,u,Du) = |\Delta u|^{p-2}\,Du$$

So the equation becomes

$$-\operatorname{div}\,|\Delta u|^{p-2}\,Du = \Delta_p u$$

The QES system has been solved in different spaces
- Spaces with a fixed exponent, as in the work [1].
- Variable exponent spaces, as in the work [6].

We are interested in the comparison of the two demonstrations. We start by introducing the functional spaces of Lebesgue and Sobolev.

1. Functional spaces
In this part the study of functional spaces with fixed exponent is extracted from the Book [?]. On the other hand, the study of variable exponent spaces is extracted from the book [5]. We begin by introducing the following notations:

— P (Ω) = {f: Ω −→ R, measurable}
— P+(Ω) = {p: Ω −→ [1.+∞).measurable}

— Ωp = Ω1 = {x ∈ Ω, p(x) = 1}
— Ωp = Ω = {x ∈ Ω: p(x) = ∞}
— Ωp∞= Ω ∞= Ω ∩ (Ω ∪ Ω) c

We note the spaces with fixed exponents
— Lebesgue Spaces Lp with p R and 1 p < .
— Sobolev spaces W m,p with m 1, p R and 1 p < .

And the spaces with variable Exposures:
— Lebesgue Spaces Lp(x) with p: Ω → [1.+∞), measurable.
— Sobolev spaces W m,p(x) with m ≥ 1 and p: Ω → [1.+∞), measurable.

Below is a table that summarizes the properties of Lebesgue spaces with a fixed exponent, denoted L^p and that with a variable exponent, denoted $L^{p(x)}$.
Many properties of fixed-exponent spaces remain true in the variable-exponent case. We cite the existence of the conjugate function, Young's Inequality, generalized Holder's Inequality, norm convergence and modular convergence. Also the topological properties such as completeness and duality remain true by associating the appropriate injections in $L^{p(x)}$.

The same applies to Sobolev's spaces. The fixed-exponent spaces $W^{m,p}$ are defined by:

Table 1: Lebesgue spaces with fixed and variable exponents.

	L*p*	**L*p*(*x*)**
Modular Functional	ρ_p(f) = ∫Ω |f|*p* dx	associated with p(x) : ρ_p(f) = ∫Ω Ω |f(x)|*p*(*x*) dx + ∩ (*c*) ∞ ess$_{\Omega0}$ sup |f x |
Norm	‖f‖ = ∫ |f|*p* 1 *p* *p*	Luxembourg norm ‖f‖$_{p(x)}$ = inf ,λ > 0 : ρ_p λ ≤ 1, *f*
Definition	Soit p ∈ R L^p(Ω = {f ∈ P (Ω) tq ‖f‖$_p$ < +∞	Soit p ∈ P$^+$(Ω). $L^{p(x)}$(Ω) = {f ∈ P (Ω) tq ∃λ > 0 : ρ_p(λf) < +∞

W m,p = {f ∈ Lp(Ω)such as Dαf ∈ Lp(Ω), |α| ≤ m}
The associated norm is ‖f‖ = Σ ‖Dαf‖. Another known space is the one denoted $W^{m,p}$. This is

Variable exponent Sobolev spaces $W^{m,p(x)}$ is defined by:
$W^{m,p}$(x) =,f ∈ $L^{p(x)}$ (Ω)such as Dαf ∈ $L^{p(x)}$(Ω), |α| ≤ m,

The norm is : $\|f\|_{m,p(x)} = \sum_{|\alpha|\le m} \|D^\alpha f\|_{m,p(x)}$ .Finally we introduce $W_0^{m,p(x)}$ is the closure of $C_0^\infty(\Omega)$ in the
() = Σ

Space $W^{m,p(x)}$(Ω). Similarly, several properties are easily transferred from spaces with fixed exposents to those with variable exposents, such as injections into Sobolev spaces. We introduce the following definitions:
Definition 1 Une fonction σ(x, u, F) is of Caratheodory if: x −→ σ(x, u, F) is measurable
∀(u, F) ∈ Rm × M m×n et (u, F) −→ σ(x, u, F) is continuous ∀x ∈ Ω.
Definition 2 The application F ›→ σ(x, u, F) is said to be monotonic if
(σ(x, u, F) − σ(x, u, G)): (F − G) ≥ 0 for all x ∈ Ω, u ∈ Rm et F, G ∈ M m×n.
Definition 3 The application F ›→ σ(x, u, F) is said to be strictly monotonic if (σ(x, u, F) − σ(x, u, G)): (F − G) = 0 drive to F = G.

2. Convergence theorems

In spaces with a fixed exponent, the convergence theorem is extracted from [2]. The second member of the system QES, v(x) is found in $W^{-1,p'}$ (Ω, R^m). Recall that:

$$\begin{aligned} -\mathrm{div}\sigma(x,u,Du) &= v(x) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \qquad (2)$$

where v traverses the dual space $W^{-1,p'}$ (Ω, R^m) of $W^{1,p}$(Ω, R^m) and σ checks the following conditions for a certain p ∈ (0, ∞):
Hypothesis 1 Continuity: σ: Ω × R^m × $M^{m\times n}$ −→ $M^{m\times n}$ is a Caratheodory function, see definition 1.

Hypothesis 2 *Growth and coercivity : It exists* $c_1 \geq 0$, $c_2 > 0$, $\lambda_1 \in L^{p'}(\Omega)$, $\lambda_2 \in L^1(\Omega)$, $0 < \alpha < p$, $\lambda_3 \in L^{(\frac{p}{\alpha})}(\Omega)$ *and* $0 < \beta \leq \frac{n}{n-p}(p-1)$ *such as :*

$$|\sigma(x,u,F)| \leq \lambda_1(x) + c_1(|u|^{\beta} + |F|^{p-1})$$

$$\sigma(x,u,F) : F \geq -\lambda_2(x) - \lambda_3(x)\,|u|^{\alpha} + c_2\,|F|^{p}$$

Hypothesis 3 Monotony: σ satisfies one of the following conditions:

1. For all x ∈ Ω and all u ∈ Rm, the application F ›⟶ σ(x, u, F) is of C^1 and is monotonous, see definition 2.
2. it exist a function W : Ω × Rm × $M^{m\times n}$ ⟶ R such as σ(x, u, F) = ∂W (x, u, F) and F ›⟶ W (x, u, F) is convex and C^1. ∂F
3. For all x ∈ Ω and all u ∈ Rm, the application F ›⟶ σ(x, u, F) is strictly monotonous, see definition3.
4. σ(x, u, F) is strictly p-quasi-monotonic in F.

Definition 4 Definition of weak solution: It is said that u: Ω −⟶ R^n is a weak solution of u ∈ W 1,p(Ω, R^m) under the hypothesis 1 - 3.

$$\begin{cases} -div\,a(x,u,Du) + b(x,u,Du) & = & v(x) & in\ \Omega \\ u & = & 0 & on\ \partial\Omega \end{cases} \qquad (3)$$

Where v traverses the dual space W −1,p*j* (Ω, Rm) of W 1,p(Ω, Rm),if
1 *u is in* $W^{1,1}(\Omega, R^m)$
2. a(., u(.), Du(.)) *is in* $L^1(\Omega, M^{m\times n})$
3. b(., u(.), Du(.)) *is in* $L^1(\Omega, R^m)$
4. *l'ation*

$$\int_{\Omega} a(x,u(x),Du(x)) : D\varphi(x)dx + \int_{\Omega} b(x,u(x),Du(x)).\varphi(x)dx = < v, \varphi >$$

is true for any φ *in* $C_c^{\infty}(\Omega, R^m)$

Theorem 2: If σ satisfies the assumptions 1 - 3 so the Dirichlet problem (QES) has a weak solution u ∈ W01,p(Ω,Rm) for all f ∈ W −1,p(Ω,Rm).
Example 3 Example : p-Laplacien. the p-Laplace operator Δp :

$$\sigma(x,u,Du) = |\Delta u|^{p-2}\,Du$$

So the equation becomes

$$-div\,|\Delta u|^{p-2}\,Du = \Delta_p u$$

The assumptions 1 - 3 are satisfied with in (b) W (x,u,F) = p1 |F|p. The p-Laplace operator is uniformly monotonic and has more general properties than the functions satisfying the hypotheses 1 - 3.

Example 4 Example: the potential W(x,u,F) is convex. We consider the corresponding elliptic problem QES : σ(x,u,F) = ∂∂WF (x,u,F). each simple method is treated differently ; the problem arises at the level of the gradients of the approximate solutions.

In their article [2], F. Augsburger et N. Hungerbuhler use the Galerkin approximation. They fix a sequence of nested vector subspaces such that their meeting is dense in W01,p(Ω,Rm). Then they write the variational formulation. After multiplying by a test function w de W01,p(Ω,Rm), we integrate on Ω, we apply the Stokes formula, and we obtain the following variational formulation:

$$\int_{\Omega} \sigma(x,u(x),Du(x)) : Dwdx \quad = \quad < f, w > \qquad (4)$$

The problem becomes: find a zero of the operator F thus defines:

$$\begin{cases} F : & W_0^{1,p}(\Omega) & \to & W^{-1,p'}(\Omega) \\ & u & \mapsto & (w \mapsto \int_{\Omega} \sigma(x,u(x),Du(x)) : Dwdx - < f, w >) \end{cases} \qquad (5)$$

Tools for the proof of the theorem 2

Properties of F

- F (u) is well defined; thanks to the growth in the hypothesis 2.
- F is linear
- F is bounded; thanks to the growth in the hypothesis 2.
- The restriction of F to a finite-dimensional vector subspace of W01,p(Ω,Rm) is continous ; thanks to the assumptions 1 and 2.

The operator G

Let k be fixed, Vk a vector subspace of dimension r admitting for base : φ1,...,φr,the discrete projection operator of F is defined on Vk:

$$\begin{cases} G: & \mathbb{R}^r \to \mathbb{R}^r \\ & (a^i) \mapsto (< F(a^j \varphi_j), \varphi_i >) \end{cases} \tag{6}$$

Properties of G

- G is continuous.
- The coercivity in the hypothesis 1 asserts,

$$G(a).a = (F(u), u) \to \infty \text{ when } \|a\|_{\mathbb{R}^n} \to \infty$$

Thus, there exists R > 0 such that

$$\forall a \in \partial B_R(0) \subset \mathbb{R}^r$$

Hence

$$G(a).a > 0$$

By a topology argument G(x) = 0 has a solution in BR (0). Therefore, for any k, it exists uk ∈ Vk such that

$$< F(u_k), v >= 0 \ , \ \forall v \in V_k$$

Stages of the demonstration

- according to the assertion of coercivity in the hypothesis 1,

$$\exists R > 0 \text{ checking } < F(u), u >> 1 \text{ when } \|u\|_{W_0^{1,p}(\Omega)} > R$$

- following Galerkin's approximations uk ∈ Vk is uniformly bounded:

$$\|u_k\|_{W_0^{1,p}(\Omega)} \leq R \text{ for all } k$$

- We can extract a sub-sequence; denoted also uk; such that uk ⇀ u is in W01,p(Ω), converges weakly in measurement and in Ls(Ω), ∀s < p∗.
- The sequence of gradients Duk generates the Young measure νx
- As uk converges to u then (uk,Duk) generates the Young measure δu(x) ⊗ νx
- For all x ∈ Ω, νx is a probability measure. You can check out its properties in the article [2].

The limit

The proof of the theorem has 4 sub-cases which depend on the hypothesis 3.

Cas iv

- We show by the absurd that the constructed Young measure is a Dirac measure p.p x ∈ Ω: νx = δDu(x).
- ⇒ Duk → Du in measure when k → ∞.
- ⇒ σ(x,uk,Duk) → σ(x,u,Du) p.p.
- Vitali's theorem. see the hypotheses of the theorem in the article [2].

$$\Rightarrow \ \sigma(x, u_k, Du_k) \to \sigma(x, u, Du)$$

In L1 (Ω)

- The convergence in L1(Ω) implies < F (u),v >= 0 for all v ∈ ∪k∈N Vk.
- By an argument of density F(u) = 0 in W01,p(Ω).

Case III

We show, thanks to the strict monotony and an inequality; see article [2], that νx = δDu(x) p.p x ∈ Ω and we join the case (iv).

Cas II

Stages of the demonstration:

$spt(\nu_x) \subset K_x = \{\lambda \in M^{m\times n} : W(x,u,\lambda) = W(x,u,Du) + \sigma(x,u,Du) : (\lambda - Du)\}$

it is shown that $\sigma(x,u,\lambda) = \sigma(x,u,Du)\ \forall \lambda \in K_x \supset spt(\nu_x)$

we define $\bar{\sigma} := \int_{M^{m\times n}} \sigma(x,u,\lambda) d\nu_x(\lambda)$ and we show that $\bar{\sigma} = \sigma(x,u,Du)$

we define the function

$$g(x,u,p) = |\sigma(x,u,p) - \bar{\sigma}(x)|$$

It is a function of Caratheodory.

we have $g_k(x) = g(x, u_k(x), Du_k(x))$ is equi-integrable and converges weakly in $L^1(\Omega)$ to $\bar{g}$.

$$\begin{aligned} \bar{g} &= \int_{R^m \times M^{m\times n}} |\sigma(x,\eta,p) - \bar{\sigma}(x)|\, d\delta_{u(x)}(\eta) \otimes d\nu_x(\lambda) \\ &= \int_{spt(\nu_x)} |\sigma(x,\eta,p) - \bar{\sigma}(x)|\ d\nu_x(\lambda) \\ &= 0 \end{aligned} \tag{7}$$

- Such that gk ≥ 0 we have gk converges strongly in L1(Ω)
- We go to the limit, which ends the demonstration.

Cas I

∀x ∈ Ω, ∀μ ∈ Mm×n in the support of νx:

$$\sigma(x,u,\lambda) : \mu = \sigma(x,u,Du) : \mu + (\nabla\sigma(x,u,Du)\mu) : (Du - \lambda)$$

we show that

$$-\sigma(x,u,\lambda) : (t\mu) \geq t((\nabla\sigma(x,u,Du)\mu)(\lambda - Du) - \sigma(x,u,Du) : \mu) + \circ(t)$$

For all t.

- σ(x,uk,Duk) is equi-integrable, it admits a weak limit in L1 notσ given by:

$$\bar{\sigma} = \int_{spt\nu_x} \sigma(x,u,\lambda) d\nu_x(\lambda) = \sigma(x,u,Du)$$

The demonstration is over

In a work published in 2014, the convergence theorem in a variable exponent space is developed in the article of [6]. Recall the quasi-linear system QES:

$$\begin{cases} -div\sigma(x,u,Du) &= v(x) \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{8}$$

Theorem 5: If σ satisfies the assumptions in the table 2 then the Dirichlet problem 8has a weak solution

$$u \in W_0^{1,p(x)}(\Omega, \mathbb{R}^n)$$

For all v in the dual of W01,p(x)(Ω,Rn).

The authors share the demonstration in several sub-lemmas. Below is a table that summarizes the different hypotheses that must be verified

$$\sigma : \Omega \times \mathbb{R}^m \times M^{m\times n} \longrightarrow M^{m\times n}$$

In order to ensure the existence of the solution. The fixed case is extracted from the article of F. Augsburger et N. Hungerbuhler [2] and the variable case of the article of F.Yongqiang et Y.Miaomiao [6].
Globally these are the same assumptions, there is a change at the level of (H2) where we remove the existence of λ3 in the case with variable exponent and a change at the level of the inequality of q(x). Else, the case (iv) of (H3) is not addressed, I wonder if it still constitutes an open question.

Table 2: Comparison of the hypotheses of the convergency theorems.

Hyptheses	fixed exponent	variable exponent
v(x)	W $^{-1,p}$⁄(Ω, R*m*)	dual of W $^{1,p(x)}$(Ω, R*m*) 0
H(1) : Continuity	σ is a function of Carathory	σ is a function of Carathory
H(2) : Growth and coercivity	It exists c1 ≥ 0, c2 > 0 λ1 ∈ L^{p}⁄(Ω) λ2 ∈ L^{1}(Ω) 0 < α < p et λ3 ∈ L(p)⁄(Ω) *α* 0 < q ≤ n (p − 1) *n−p* vfy (x, u, F)| ≤ | σ λ1(x) +c1(|u|*β* +|F|*p*−1) et σ(x, u, F) : F ≥ −λ2(x) − λ3(x) |u|*α* +c2 |F|*p*	It exists c1 ≥ 0, c2 > 0 λ1 ∈ L^{p}⁄(*x*)(Ω) λ2 ∈ L^{1}(Ω) *p*(*x*)−1 < q(x) ≤ *n* (p(x) − 1) *p*(*x*) *n−p*(*x*) vfy |σ(x, u, F)| ≤ λ1(x) +c1(|u|*q*(*x*) +|F|*p*(*x*)−1) et σ(x, u, F) : F ≥ −λ2(x) +c2 |F|*p*(*x*)
H(3) : Monotony	σ(x, u, F) vfy (i) or (ii) or (iii) or (iv)	σ(x, u, F) vfy (i) or (ii) or (iii)
	(i)∀x ∈ Ω, ∀u ∈ R*m* F ›→ σ(x, u, F) is C^{1} and is monotonous	(i) ∀x ∈ Ω, ∀u ∈ R*m* F ›→ σ(x, u, F) is C^{1} and is monotonous
	(ii)∃W : Ω × R*m* × M $^{m×n}$ → R checking σ(x, u, F) = ∂*W* (x, u, F) ∂*F* and F ›→ W (x, u, F) is convex and C^{1}	(ii)∃W : Ω × R*m* × M $^{m×n}$ → R checking σ(x, u, F) = ∂*W* (x, u, F) ∂*F* and F ›→ W (x, u, F) is convex and C^{1}
	(iii) F ›→ σ(x, u, F) is strictly monotonous	(iii)F ›→ σ(x, u, F) is strictly monotonous
	(iv)σ(x, u, F) is strictly p-quasi-monotonous in F	not mentioned

Tools for the proof of the theorem
In the following, we define the functional

$$J(u) : \omega \mapsto \int_{\Omega} \sigma(x, u(x), Du(x)) : D\omega \, dx - < f, \omega > \tag{9}$$

For all u ∈ W01,p(x), the functional J(u) is linear and bounded. Moreover, the restriction of J to a finite linear subspace of W01,p(x) is continous.
We note:

$$p'(x) = \begin{cases} \infty & \text{si} \quad x \in \Omega_1 = \{x \in \Omega, p(x) = 1\} \\ 1 & \text{si} \quad x \in \Omega_\infty = \{x \in \Omega, p(x) = \infty\} \\ \frac{p(x)}{p(x)-1} & \text{else} \end{cases} \tag{10}$$

The authors use the Galerkin approximation. Either

$$V_1 \subset V_2 \subset ... \subset W_0^{1,p(x)}(\Omega)$$

A sequence of finite-dimensional subspaces with the property ∪i∈NVi is dense in W01,p(x). For any k, it is assumed that the dimension of Vk is rk and Φ1,...,Φrk is a base. We note Pri=1 aiΦi = aiΦi then we define the discrete.
projection operator:

$$G \quad : \quad r \to r \\ \left(a^i\right)_{1\leq i\leq r} \mapsto \left(< J\left(a^i\Phi_i\right), \Phi_j >\right)_{1\leq j\leq r} \tag{11}$$

Lemma 1 The application of discrete projection G on Vk is continuous. In addition,

$$G\left(a\right).a \to \infty$$

when ‖a‖ → ∞where . means the dot product of two vectors in Rr.

Lemma 2: For all k ∈ N, it exists uk ∈ Vk such that

$$< J\left(u_k\right), v >= 0 \ , \ v \in V_k$$

Lemma 3: If uk ⇀ u in W01,p(x) and σ satisfies the assumptions of the tables 2 (H1) − (H3), then for all v ∈ W01,p(x), we have:

$$\int_\Omega \left(\sigma(x, u_k(x), Du_k(x)) : D\omega - \sigma(x, u(x), Du(x)) : D\omega\right) dx \to 0$$

when k → ∞

Definition 5: fk converges weakly to t if and only if

$$< g^*, f_k > \to < g^*, f >$$

For all g∗ in the dual of W01, p(x).

Demonstration 6: The demonstration is based on a reasoning by absurd. Thus, it suffices to show that there exists u ∈ W01, p(x) such that for any ω ∈ W01,p(x) we have < J(u),v >= 0. For the entire demonstration you can consult the article [6].

Remark 1: In the article [6], the authors wrote: it is sufficient to prove that for any v ∈ W01,p(x), there exists u ∈ W01,p(x) such that < J(u),v >= 0. I think it was necessary to write: there exists u ∈ W01,p(x) such that for any v ∈ W01,p(x) we have < J(u),v >= 0.

3 Conclusion

As explained in the introduction, the author treats an elliptic problem in two functional spaces. The first is the fixed-exponent Sobolev space, the second is the variable-exponent Sobolev space. The weak convergence theorem are reported but the demonstrations are partial.

4. References

1. F Augsburger. Young measures and quasilinear systems in divergence form with weak monotonicity, University of Fribourg; c2004.
2. F Augsburger et N. Hungerbuhler, Quasilinear elliptiC systems in divergenCe form with weak monotoni City and nonlinear physiCal data, Electronic Journal of Differential Equations; c2004.
3. L Diening. Introduction to applied mathematics, European University Editions; c2012. ISBN: 978-3-8417-8154-3.
4. Y Annabi. Introduction to linear operators and nonlinear operators, Book, European University Editions; c2016, ISBN: 978-3-639-50779-9.
5. Y Annabi, Lebesgue and Sobolev spaces with variable exponents, Springer; c2010.
6. Y Fu, M Yang. Existence of solutions for quasilinear elliptic systems in divergence form with variable growth, Journal of Inequalities and Applications; c2014.