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Expansion mapping in controlled metric space and extended B-metric space

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Abstract

This paper delves into the intricate study of expansion mappings within the frameworks of controlled metric spaces and extended B-metric spaces. Expansion mappings, known for their crucial role in fixed-point theory and iterative processes, are examined under the lens of these generalized metric spaces to uncover their distinct properties and extended applicability. Controlled metric spaces, which incorporate a dynamic control function to modulate the distance measurements, offer a refined approach to traditional metric space concepts. This flexibility allows for a more nuanced understanding of spatial relationships and convergence behaviors. Extended B-metric spaces, by relaxing the stringent requirements of the triangle inequality, open new avenues for theoretical exploration and practical application, accommodating broader classes of functions and sequences. In this study, we aim to provide a comprehensive analysis of the behavior of expansion mappings in these sophisticated metric frameworks. We will present new theoretical results that extend and generalize existing principles, highlighting the interplay between the control functions in controlled metric spaces and the relaxed conditions in extended B-metric spaces. Additionally, we explore practical applications of these findings in various fields, including optimization, computational mathematics, and the analysis of iterative methods. By bridging the gap between classical metric spaces and their generalized counterparts, this paper contributes to a deeper understanding of the mathematical foundations and potential applications of expansion mappings. The results presented herein pave the way for further research and development in this vibrant area of mathematical analysis.

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1. Introduction

In the realm of mathematical analysis, metric spaces serve as the cornerstone for various theoretical and applied studies. They provide a structured environment for understanding and measuring distances, which is crucial for analyzing convergence, continuity, and other fundamental concepts. However, traditional metric spaces, with their rigid distance functions, may not be sufficiently adaptable for certain complex scenarios. This limitation has led to the development of more generalized metric spaces, such as controlled metric spaces and extended B-metric spaces, which offer enhanced flexibility and broader applicability.

Controlled Metric Spaces

Controlled metric spaces are an evolution of conventional metric spaces, introduced to incorporate a control function that dynamically adjusts the distance between points. This control function, often context-dependent, allows for a more refined analysis of spatial relationships. By modulating the distance based on specific criteria, controlled metric spaces enable the study of systems where traditional metrics fall short, such as in scenarios with varying or context-specific distance measures.

This flexibility is particularly beneficial in applications involving dynamic or adaptive systems, where fixed metrics may not adequately capture the intricacies of the environment.

Extended B-Metric Spaces

Extended B-metric spaces, on the other hand, relax the classical triangle inequality condition of metric spaces. In an extended B-metric space, the distance function satisfies a generalized form of the triangle inequality, which broadens the scope of spaces that can be analyzed. This relaxation allows for the inclusion of a wider array of functions and sequences, thereby providing a more versatile framework for various mathematical explorations. The extended B-metric spaces have proven to be especially useful in fixed-point theory, iterative methods, and other areas where traditional metric constraints are too limiting.

Expansion Mappings

Expansion mappings, a subset of mappings that increase the distance between points, play a significant role in the analysis of controlled metric spaces and extended B-metric spaces. These mappings are fundamental in fixed-point theory, where they are used to demonstrate the existence and uniqueness of fixed points under certain conditions. In the context of controlled metric spaces, expansion mappings interact with the control function to provide deeper insights into the behavior of iterative processes and convergence. Similarly, in extended B-metric spaces, they facilitate the study of more complex systems by leveraging the relaxed triangle inequality to explore new theoretical results and applications.The objective of this paper is to explore the properties and applications of expansion mappings within the frameworks of controlled metric spaces and extended B-metric spaces. By examining these mappings in detail, we aim to extend existing theories and introduce new perspectives on their utility in both theoretical and practical contexts. Through this comprehensive analysis, we hope to contribute to the broader understanding of metric space generalizations and their potential to address complex mathematical and real-world challenges.

2. Preliminaries

Definition 1. ([7]) Let X be a nonempty set and $\theta : X \times X \longrightarrow [1,\infty)$. A function $d\theta : X \times X \longrightarrow [0,\infty)$ is called an extended b-metric space if, for all $x,y,z \in X$, it satisfies the following properties:

- $d\theta(x,y) = 0$ iff $x = y$;
- $d\theta(x,y) = d\theta(y,x)$;
- $d\theta(x,z) \leq \theta(x,z)[d\theta(x,y) + d\theta(y,z)]$.

The pair $(X,d\theta)$ is called an extended b-metric space.

Remark 2. If we take $\theta(x,y) = s$ for $s \geq 1$, then we obtain the definition of a b- metric space.

Example 3. *Let* $X = \{2,3,4\}$. *Define* $\theta : X \times X \longrightarrow [1,\infty)$ *and* $d\theta : X \times X \longrightarrow [0,\infty)$ as follows:

$\theta(x,y) = 2 + x + y$, $d\theta(2,2) =$

$d\theta(3,3) = d\theta(4,4) = 0$,

$d\theta(2,3) = d\theta(3,2) = 30, d\theta(2,4) = d\theta(4,2) = 200, d\theta(3,4) = d\theta(4,3) = 2000$.

Then $(X,d\theta)$ is an extended b-metric space.

The notions of a Cauchy sequence and a convergent sequence in extended
b-metric spaces are defined as follows:

Definition 4. ([7]) Let $(X,d\theta)$ be an extended b-metric space and (xn) be a sequence in X.

1. A sequence (xn) in X is said to converge to $x \in X$ if, for every $\varrho > 0$, there exists $N\varrho \in \mathbb{N}$ such that $d\theta(xn,x) < \varrho$ for all $n \geq N\varrho$. In this case, we write $\lim n\to\infty\ xn = x$.
2. A sequence (xn) in X is said to be Cauchy if, for every $\varrho > 0$, there exists $N\varrho \in \mathbb{N}$ such that $d\theta(xn,xm) < \varrho$ for all $n,m \geq N\varrho$.

An extended b-metric space $(X,d\theta)$ is said to be complete if every Cauchy sequence in X is convergent.

Example 5. *[7] Let* $X = C([a,b],\mathbb{R})$ *be the space of all continuous real valued functions define on* $[a,b]$. *X is a complete extended b-metric space by considering* $d\theta(x,y) = \sup |x(t) - y(t)|2,\ t\in[a,b]$

with $\theta(x,y) = |x(t)| + |y(t)| + 2$, *where* $\theta : X \times X \longrightarrow [1,\infty)$.

Definition 6. ([18]) Let A and B be two nonempty subsets of a space X. A mapping $T : A \cup B \to A \cup B$ is called cyclic if $T(A) \subset B$ and $T(B) \subset A$.

Definition 7. [1] A continuous function $F : [0,\infty)2 \to \mathbb{R}$ is called an C-class function if it satisfies:

1. $F(s,t) \leq s$,
2. $F(s,t) = s$ implies that either $s = 0$ or $t = 0$ for all $s,t \in [0,\infty)$.

We denote the set of C-class by C.

Example 8. ([1]) For s,t ∈ [0,∞), define the functions F : [0,∞)2 → R by

- F(s,t) = s − t.
- F(s,t) = αs for some α ∈ (0,1).
- F(s,t) = s/(1 + t)r for some r ∈ (0,∞).
- F(s,t) = (s − t)/(1 + t).

Then these functions are elements of C.

Definition 9. ([8]) Let ψ : [0,∞) → [0,∞) be a continuous, nondecreasing function. Then ψ is called an altering distance function if ψ(t) = 0 ⇔ t = 0.

We denote the set of altering distance functions by Φa.

Definition 10. ([1]) Let ϕ : [0,∞) → [0,∞) be a continuous mapping. Then ϕ
is called an ultra altering distance function if ϕ(t) > 0 for all t > 0. We denote the set of ultra altering distance functions by Φu.

3. Main Result

In this section, we utilize the notion of C-class functions to introduce some fixed point results for a cyclic mapping.

Theorem 11. Let (X,dθ) be a complete extended b-metric space. Let A and B be two nonempty closed subsets of X such that A∩B 6= ∅ and X = A∪B. Let f : A ∪ B → A ∪ B be a cyclic mapping. Suppose that there exist F ∈ C, ψ ∈ Φa and φ ∈ Φu such that:

ψ(dθ(fx,fy)) ≤ F(ψ(dθ(x,y)),φ(dθ(x,y))) forallx,y ∈ X. (1)

Assume for x0 ∈ X, we have

Lim θ(xn,xm) = 1, (2)

n,m→∞

where xj = fjx0. If f is continuous, then f has a unique fixed point in A ∩ B.

Proof. Let x0 ∈ A. Then x1 = fx0 ∈ B and x2 = fx1 ∈ A. Continuing this process, we obtain a sequence (xn) in X, such that fxn = xn+1 with x2n ∈ A and x2n+1 ∈ B for all n ∈ N.

First, we want to show that: limn→∞ dθ(xn,xn+1) = 0. Let n ∈ N,

ψ(dθ(xn,xn+1)) = ψ(dθ(fxn−1,fxn))

≤ F(ψ(dθ(xn−1,xn)),φ(dθ(xn−1,xn)))

≤ ψ(dθ(xn−1,xn)). (3)

Since ψ ∈ Φa, then we have

dθ(xn,xn+1) ≤ dθ(xn−1,xn) for all n ∈ N. (4)

This shows that (dθ(xn,xn+1)) is decreasing. Then, there exists some r ≥ 0 such that lim dθ(xn,xn+1) = r.

n→∞ (5)

Assume that r > 0. Letting n → ∞ in (3). Since F, ψ and ϕ are continuous, we get

ψ(r) ≤ F(ψ(r),φ(r)). (6)

So ψ(r) = 0 or φ(r) = 0. Since φ(r) > 0, we have r = 0, which is a contradiction. Hence, r = 0 and so lim dθ(xn,xn+1) = 0. (7)
n→∞

Now, we want to show that (xn) is a Cauchy sequence. Assume (xn) is not a Cauchy sequence. Then, there exists ϱ > 0 and subsequences (xnk) and (xmk) of (xn) with nk > mk > k such that

$d\theta(x_{nk},x_{mk}) \geq \varrho.$ (8)

Further, corresponding to mk, we can choose nk in such a way that it is the smallest integer with nk > mk and satisfying (8). So

$d\theta(x_{nk-1},x_{mk}) < \varrho.$ (9)

Then we have

$0 < \varrho \leq d\theta(x_{nk},x_{mk}) \leq \theta(x_{nk},x_{mk})[d\theta(x_{nk},x_{nk-1}) + d\theta(x_{nk-1},x_{mk})]$

$\leq \theta(x_{nk},x_{mk})[d\theta(x_{nk},x_{nk-1}) + \varrho].$ (10)

Letting $k \to \infty$ and using (7) and (2), we get

$\lim_{k\to\infty} d\theta(x_{nk},x_{mk}) = 0$ (11)

Also

$d\theta(x_{nk},x_{mk}) \leq \theta(x_{nk},x_{mk})d\theta(x_{nk},x_{nk-1}) + \theta(x_{nk},x_{mk})\theta(x_{nk-1},x_{mk})\,[d\theta(x_{nk-1},x_{mk-1}) + d\theta(x_{mk-1},x_{mk})]$
$\leq \theta(x_{nk},x_{mk})d\theta(x_{nk},x_{nk-1}) + \theta(x_{nk},x_{mk})\theta(x_{nk-1},x_{mk})\,\theta(x_{nk-1},x_{mk-1})d\theta(x_{nk-1},x_{nk}) + \theta(x_{nk},x_{mk})\theta(x_{nk-1},x_{mk})\theta(x_{nk-1},x_{mk-1})\theta(x_{nk},x_{mk-1})d\theta(x_{nk},x_{mk})$

$+ \theta(x_{nk},x_{mk})\theta(x_{nk-1},x_{mk})\theta(x_{nk-1},x_{mk-1})\,\theta(x_{nk},x_{mk-1})d\theta(x_{mk},x_{mk-1}) + \theta(x_{nk},x_{mk})\theta(x_{nk-1},x_{mk})\,d\theta(x_{mk-1},x_{mk}).$ (12)

As $k \to \infty$ in the above inequalities and using (2), (7) and (11), we get:

$\lim_{k\to\infty} d\theta(x_{nk-1},x_{mk-1}) = \varrho.$ (13)

Then by (1), we have

$\psi(d\theta(x_{nk},x_{mk})) \leq F(\psi(d\theta(x_{nk-1},x_{mk-1})),\varphi(d\theta(x_{nk-1},x_{mk-1})))$

$\leq \psi(d\theta(x_{nk-1},x_{mk-1})).$ (14)

Letting $k \to \infty$, we obtain

$\psi(\varrho) \leq F(\psi(\varrho),\varphi(\varrho)) \leq \psi(\varrho).$ (15)

So $\psi(\varrho) = 0$ or $\varphi(\varrho) = 0$. This implies that $\varrho = 0$, which is a contradiction.

Hence, (x_n) is a Cauchy sequence. So there exists $u \in X$ such that $\lim_{n\to\infty} x_n = u$.

Since f is continuous, we have: $\lim_{n\to\infty} fx_n = fu$. On the other hand: $\lim_{n\to\infty} fx_n$

$= \lim_{n\to\infty} x_{n+1} = u$, then by uniqueness of the limit, we have $fu = u$.

Since $(x_{2n}) \in A$ and A is closed, we have $u \in A$. Also, since $(x_{2n+1}) \in B$ and B is closed, we have $u \in B$. Hence, u is a fixed point of f in $A \cap B$.

To prove the uniqueness of u, we assume there exists $v \in X$ such that $fv = v$.

Then by (1), we have

$\psi(d\theta(u,v)) = \psi(d\theta(fu,fv)) \leq F(\psi(d\theta(u,v)),\varphi(d\theta(u,v)))\leq \psi(d\theta(u,v)).$ (16)

So, $\psi(d\theta(u,v)) = 0$ or $\varphi(d\theta(u,v)) = 0$. This implies that $d\theta(u,v) = 0$. Hence $u = v$. Thus, f has a unique fixed point in $A \cap B$. □
By choosing $A = B = X$ in Theorem 11 we get the following result:

Corollary 12. Let $(X,d\theta)$ be a complete extended b-metric space, and let $f : X \to$

X be a mapping. Suppose that there exist F ∈ C, φ ∈ Φa and φ ∈ Φu such that:

$$\psi(d\theta(fx,fy)) \le F(\psi(d\theta(x,y)),\varphi(d\theta(x,y))) \text{ for all } x,y \in X. \quad (17)$$

Assume for $x_0 \in X$, we have
Lim θ(xn,xm) = 1,

$$n,m \to \infty \quad (18)$$

where xj = fjx0. If f is continuous, then f has a unique fixed point in X.

Corollary 13. Let (X,dθ) be a complete extended b-metric space, and let f be a self mapping defined on X satisfying
$(1 + \phi(d\theta(x,y)))\psi(d\theta(fx,fy)) \le \psi(d\theta(x,y)) - \phi(d\theta(x,y))$ for all x,y ∈

X, where ϕ ∈ Φu and ψ ∈ Φa. Assume for x0 ∈ X, we have

Lim θ(xn,xm) = 1, n,m→∞

where xj = fjx0. If f is continuous, then f has a unique fixed point in X.

Proof. Define F by F(s,t) = (s−t)/(1+t), the result follows from

Theorem 11.
Corollary 14. Let (X,dθ) be a complete extended b-metric space and f : A ∪B → A∪B be a cyclic mapping. Let A and B be two nonempty closed subsets of X such that A ∩ B 6= ∅ and X = A ∪ B. Assume that there exist ψ ∈ Φa and α ∈ [0,1) such that

$$\psi(d\theta(fx,fy)) \le \alpha\psi(d\theta(x,y)) \text{ for all } x,y \in X. \quad (20)$$

Assume for x0 ∈ X, we have
lim θ(xn,xm) = 1, (21)

n,m→∞

where xj = fjx0. If f is continuous, then f has a unique fixed point in A ∩ B. Proof.

Define F by F(s,t) = αs, the result follows from Theorem 11.

Corollary 15. Let (X,d) be a complete metric space and let f : X → X be a mapping. Suppose that there exist F ∈ C, ψ ∈ Φa and

$$\varphi \in \Phi u \text{ such that } \psi(d(fx,fy)) \le F(\psi(d(x,y)),\varphi(d(x,y))) \text{ for all } x,y \in X. \quad (22)$$

If f is continuous, then f has a unique fixed point in X.

Example 16. Let X = [−1,1]. Define dθ : X × X → R^+ by dθ(x,y) = |x − y| and θ : X × X → [1,∞) by θ(x,y) = |x| + |y| + 1. Let A = [−1,0], B = [0,1], and define f : X → X by fx = −x/2. Also, define F ∈ C by F(s,t) = s − t and φ ∈ Φu by φ(t) = t/4 and ψ ∈ Φa by φ(t) = t. Then
(a) dθ is a complete extended b-metric space on X.
(b) A and B are closed subsets of X.
(c) f is continuous and cyclic.
(d) f satisfy the inequality:

ψ(dθ(fx,fy)) ≤ F(ψ(dθ(x,y)),φ(dθ(x,y))) for all x,y ∈ X.

(e) Let x0 ∈ X, we have: limn,m→∞ θ(xn,xm) = 1. To prove (d), let x,y ∈ X. Then

$$\psi(d_\theta(fx,fy)) = d_\theta(-\frac{x}{2},-\frac{y}{2}) = |-\frac{x}{2}+\frac{y}{2}| = \frac{1}{2}|x-y|$$

and

$$F(\psi(d_\theta(x,y)),\phi(d_\theta(x,y))) = d_\theta(x,y) - \phi(d_\theta(x,y)) = \frac{3}{4}|x-y|$$

So,

$\psi(d\theta(fx,fy)) \leq F(\psi(d\theta(x,y)),\varphi(d\theta(x,y)))$

Now, to prove (e), let x0 ∈ X, then xn = (−1)nx0/2n and xm = (−1)mx0/2m and

$$\lim_{n,m\to\infty} \theta(x_n, x_m) = \lim_{n,m\to\infty} (\frac{|x_0|}{2^n} + \frac{|x_0|}{2^m} + 1) = 1$$

The example satisfies all the hypotheses of Theorem 11. Hence, f has a unique fixed point in A ∩ B = {0}.

References

1. Ansari AH. Note on ϕ-ψ-contractive type mappings and related fixed point. In: The 2nd Regional Conference on Mathematics and Applications; 2014; Payame Noor University. 2014. p. 377–380.
2. Bakhtin IA. The contraction mapping principle in almost metric spaces. Funct Anal. Gos. Ped. Inst. Unianowsk. 1989;30:26–37.
3. Banach S. Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fund Math. 1922;3:133–181.
4. Bataihah A, Shatanawi W, Tallafha A. Fixed point results with simulation functions. Nonlin Funct Anal Appl. 2020;25(1):13–23.
5. Bataihah A, Tallafha A, Shatanawi W. Fixed point results with Ω−distance by utilizing simulation functions. Ital J Pure Appl Math. 2020;43:185–196.
6. Czerwik S. Contraction mappings in b-metric spaces. Acta Math Inform Univ Ostrav. 1993;1:5–11.
7. Kamran T, Samreen M, Ain QU. A generalization of b-metric space and some fixed point theorems. Mathematics. 2017;19(5):Art. 19.
8. Khan MS, Swaleh M, Sessa S. Fixed point theorems by altering distances between the points. Bull Austral Math Soc. 1984;30:1–9.
9. Mukheimer A, Mlaiki N, Abodayeh K, Shatanawi W. New theorems on extended b-metric spaces under new contractions. Nonlin Anal Model Control. 2019;24(6):870–83.
10. Qawaqneh H, Noorani MSM, Shatanawi W. Fixed point theorems for (α,k,θ)-contractive multi-valued mapping in b-metric space and applications. Int J Math Comput Sci. 2019;4:263–283.
11. Qawasmeh T, Tallafha A, Shatanawi W. Fixed point theorems through modified ω−distance and application to nontrivial equations. Axioms. 2019;8(2):Art. 57.
12. Roshan JR, Parvaneh V, Sedghi S, Shobkolaei N, Shatanawi W. Common fixed points of almost generalized (ψ,ϕ)-s-contractive mappings in ordered b-metric spaces. Fixed Point Theory Appl. 2013;2013(1):Art. 159.
13. Shatanawi W, Postolache M. Common fixed point results for mappings under nonlinear contraction of cyclic form in ordered metric spaces. Fixed Point Theory Appl. 2013;2013:Art. 60.
14. Shatanawi W. Fixed and common fixed point for mapping satisfying some nonlinear contraction in b-metric spaces. J Math Anal. 2016;7(4):1–12.
15. Shatanawi W, Abodayeh K, Mukheimer A. Some fixed point theorems in extended b-metric spaces. Sci Bull Ser A Appl Math Phys. Politeh Univ Buchar. 2018;80(4):71–78.
16. Shatanawi W, Pitea A, Lazović R. Contraction conditions using comparison functions on b-metric spaces. Fixed Point Theory Appl. 2014;2014:Art. 135.
17. Shatanawi W, Postolache M. Coincidence and fixed point results for generalized weak contractions in the sense of Berinde on partial metric spaces. Fixed Point Theory Appl. 2013;2013:Art. 54.
18. Ansari AH, Shatanawi W. C-class functions on fixed and common fixed point results for cyclic mappings of Omega-distance. Nonlin Anal Model Control. 2017;22(6):739–752.