International Journal of Multidisciplinary Research and Growth Evaluation.

Contractive Mapping in Controlled Metric Space and Extended B-Metric Spaces

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Article Info

ISSN (online): 2582-7138
Volume: 06
Issue: 01
January-February 2025
Received: 09-11-2024
Accepted: 12-12-2024
Page No: 1552-1561

Abstract
This paper discusses contractive mappings in controlled metric spaces and extended b-metric spaces. It starts with setting up the foundational definitions and properties of the mappings of contractive mappings and associated aspects emphasized in terms of their crucial role in ensuring convergence in iterative processes. Investigation is made in controlled metric spaces on how a controlled approach may increase the flexibility and applicability of contractive mappings, specifically in non-standard metrics. This paper extends the ideas developed here further to b-metric spaces where we discuss what aspects of extending the traditional metric framework are affected. Examples and applications show that such mappings have much importance in various mathematical as well as computational contexts. Results obtained contribute to a broader understanding of fixed-point theory and its potential applications in solving complex problems on different areas that include optimization and dynamical systems.

DOI: https://doi.org/10.54660/.IJMRGE.2025.6.1.1552-1561

Keywords: Contractive Mapping, Controlled Metric Space, Extended b-Metric Space, Fixed Point Theorem, Banach's Fixed Point Principle, Distance Function, Contraction Condition

Introduction

Contractive mappings form one of the most important classes of mappings in fixed-point theory. This is vital in many branches of mathematics, including analysis and topology. In classical metric spaces, one has the Banach Fixed Point Theorem, which gives necessary and sufficient conditions for the existence and uniqueness of a fixed point for the whole class of contractive mappings. Nevertheless, contractive mappings have been analyzed in even more general structures, such as controlled metric spaces and extended b-metric spaces.

Controlled metric spaces. This has a corresponding framework based on a control function allowing for more variations in the definition of distances and convergence. It is an extension of fixed-point results to certain applied situations where the classical conditions are not satisfied. Extended b-metric space. This is the extension of the concept of b-metrics that provides a richer structure within which convergence and even continuity can be studied in far greater detail. This paper is aimed to attract attention to the importance of contractive mappings within the framework of more advanced spaces: that it is useful for deep solution of complex problems and theoretical development. Investigation of properties and consequences of such mappings may be a way opening new ways within the fixed-point theory, crossing different mathematical fields of application.

Preliminaries: I

1. Controlled Metric Spaces

A controlled metric space is a generalization of a metric space where distances are measured using a control function. Formally, let X be a set and d:

$\mathbf{X} \times \mathbf{X} \to [0, \infty)$ be a function

Satisfying

- Non-negativity: d(x,y) ≥0 for all x,y **∈X.**
- Identify of indiscernibles: d(x,y)=0 if and only if x=y.y)
- Symmetry: d(x,y)=d(yx,,x) for all x,y **∈X.**
- Triangle inequality: d(x,z)≤d(x,y)+d(y,z).

2. Contractive Mappings

A function T: X→X is called a contractive mapping if there exists a constant 0≤k<1 such that:

$$d(T(x),T(y))\leq k.d(x,y)\forall x,y\in X.$$

3. Controlled Metric Spaces

Controlled metric spaces generalize traditional metric spaces by allowing a more flexible distance function, often incorporating an auxillary function to control the distance.

Definition: A controlled metric space is a pair (X,d) where d:X × X→R

Satisfies the properties of a metric, possibly modified by a control function ϕ:R+ → R+, where:

$$d(x,y)\leq\phi(d(T(x),T(y)))\ \forall x,y\in X.$$

4. Extended b-Metric Spaces

An extended b-metric space is a generalization of a b-metric space where distance between points may not satisfy the traditional triangle inequality, But instead fulfills a relaxed condition.

Definition: An extended b-metric space is defined by a function

d:X × X →[0,∞)such that for all x,y,z∈X:

- (1) d(x,y)=0 if and only if x=y.
- (2) d(x,y)=d(y,x).
- (3) d(x,z)≤d(x,y)+d(y,z)+ϕ(d(x,y),d(y,z)), where ϕ is a control function.

5. Existence of Fixed Points

In controlled metric spaces and theorems often require specific conditions on the contractive mappings, including:

- The mapping must be continuous.
- The space must be complete in a sense adapted to the control function.

6. Applications

Contractive mappings in these generalized spaces are instrumental in various fields such as:

- Nonlinear analysis.
- Mathematical Biology.
- Economics and game theory.

Conclusion

Understand contractive mappings in controlled metric spaces and extended b-metric spaces expands the scope of fixed point theory, offering richer frameworks for analysis in various mathematical disciplines

Preliminaries: II

Definition 1.1: (b-metric space) [14].

Suppose db: X × X → R+ be a map on a nonvoid set X, satisfying

(*1) db(ϑ,η) = 0 if and only if ϑ = η,

(*2) db(ϑ,η) = db(η,ϑ),

(*3) db(ϑ,η) ≤ s[db(ϑ,z)+db(z,η)],for some constant s ≥ 1 for all ϑ, η ∈ X.

Then, db is called a b-metric on X and (X,db) is referred to as a b-metric space with a constant s.

Remark 1.2 It should be noted that for s=1, b-metric space reduces to a metric space.Thus, every metric space is a b-metric space with s=1 but, in general, a b-metric space is not a metric space. This metric has been further generalized by replacing the constant s by a function. The concept of extended b-metric space is given as follows:

Definition 1.3 (Extended b-metric space) [20].

Suppose that X is a nonvoid set and bϕ*:* X × X → R+, where ϕ: X × X → [1, ∞), is function satisfying

(**1) bϕ(ϑ,η)=0 iff ϑ = η,

(**2) bϕ(ϑ,η)=bϕ(η,ϑ),

(**3) bϕ(ϑ,η)≤ϕ(ϑ,η)[bϕ(ϑ,z)+[bϕ(z,η)],, For all $\vartheta,\eta \in X$.Then, $b\phi$ is referred to as extended b-metric on X and $(X,b\phi)$ is called an extended metric space.

Remark 1.4 It should be noted that for $\phi(\vartheta,\eta)=s$, for all $\vartheta,\eta \in X$ and some constant $s \geq 1$, the extended b-metric space reduces to a b-metric space.Each b-metric space is therefore an extended b-metric space, but not the other way around.

Definition 1.5. [36]

A sequence $\{\vartheta n\}$ in an order set $(X,\preceq)$ is said to be increasing or ascending if, for all $m,n \in N0$ such that $m < n$, we have $\vartheta m \preceq \vartheta n$. It is said to be strictly increasing if $\vartheta m \preceq \vartheta n$ and $\vartheta m \neq \vartheta n$.We denote this an $\vartheta m \prec \vartheta n$.

Definition 1.6. [33]. Let $(X, d,\preceq)$ be any ordered metric space. X has the t-property if every strictly increasing Cauchy sequence $\{\vartheta n\}$ in X has a strict upper bound in X, i.e., there exists $u \in X$ such that $\vartheta n \preceq u$, for all $n \in N0$.

Next, we state the definition of the t-property on extended b-metric spaces. We refer the readers to recall the convergence and Cauchy sequence on extended b-metric spaces (Definition 4 in [20]).

Definition 1.7. An ordered extended b-metric space$(X, b\phi,\preceq)$is said to have the t-property, if every strictly increasing Cauchy sequence $\{\vartheta n\}$ in X has a strict upper bound in X, i.e., there exists $u \in X$ such that $\vartheta n \prec u$.

Example 1.8. Let X = R or X = Q or X = (a, b], a, b ∈ R be equipped with the natural ordering $\leq$ and the usual metric. Then X has the t-property.

We complete this section by presenting some examples to demonstrate the concepts defined above.

Example 1.9. Let X = Q be endowed with the metric $b\phi: X \times X \to [0, \infty)$ given by $b\phi(\vartheta, y) = |\vartheta - y|^2$, where $\phi: X\times X \to [1, \infty)$ is defined by $\phi(\vartheta,y)= \vartheta^2 + y^2 +2$. If we take the sequence $\{\vartheta n\}$ as an increasing Cauchy sequence in Q such that $\vartheta n^2< 2$, for all $n \in N$, then $\{\vartheta n\}$ is a Cauchy sequence and converges to $\sqrt{2}$. This shows that $(X, b\phi,\preceq)$is not complete but has t-property as every rational number greater than $\sqrt{2}$ is an upper bound of $\{\vartheta n\}$.

Example 1.10. Suppose $X = [0, 2] \cap Q$. We define an extended b-metric $b\phi: X \times X \to [0, \infty)$ by $b\phi(\vartheta, \eta) = 0$, if $\vartheta = \eta = \vartheta + \eta$, if $\vartheta \neq \eta$.

Let us define $\phi: X \times X \to [1, \infty)$ defined by $\phi(\vartheta, \eta) = 8 + \eta - \vartheta$.

Then clearly, $(X, b\phi)$ is not a complete extended b-metric but has the t-property.

Main Results

1. Fixed points of ordered extended b-metric space having the t-property prior to establishing the main result, it is imperative to prove the following lemma.

Lemma 1.1. Let $(X, b\phi,\preceq)$ be an ordered extended b-metric space. Then for any sequence $\{\vartheta n\}$ in X, we have

$$b\phi(\vartheta n, \vartheta m) \leq \phi(\vartheta n, \vartheta m)b\phi(\vartheta n, \vartheta n+1) + \phi(\vartheta n, \vartheta m)\phi(\vartheta n+1, \vartheta m)b\phi(\vartheta n+1, \vartheta n+2) + \cdots$$
$$+\phi(\vartheta n, \vartheta m)\phi(\vartheta n+1, \vartheta m)\phi(\vartheta n+2, \vartheta m)\cdots \phi(\vartheta m-1, \vartheta m)b\phi(\vartheta m-1, \vartheta m), \text{ for all } n, m \in N \text{ with } n < m.$$

Proof. By using the triangle inequality, we have

$$b\phi (\vartheta n, \vartheta m) \leq \phi(\vartheta n, \vartheta m)[b\phi(\vartheta n, \vartheta n+1) + b\phi(\vartheta n+1, \vartheta m)]$$
$$= \phi (\vartheta n, \vartheta m) b\phi (\vartheta n, \vartheta n+1) + \phi (\vartheta n, \vartheta m) b\phi (\vartheta n+1, \vartheta m).$$

Again using the triangle inequality, we have

$$b\phi(\vartheta n, \vartheta m) \leq \phi(\vartheta n, \vartheta m)b\phi(\vartheta n, \vartheta n+1) + \phi(\vartheta n, \vartheta m)\phi(\vartheta n+1, \vartheta m)[b\phi(\vartheta n+1, \vartheta n+2) + b\phi(\vartheta n+2, \vartheta m)]$$

Continuing in this way, we get

$$b\phi(\vartheta n, \vartheta m) \leq \phi(\vartheta n, \vartheta m)b\phi(\vartheta n, \vartheta n+1) + \phi(\vartheta n, \vartheta m)\phi(\vartheta n+1,\vartheta m)b\phi(\vartheta n+1, \vartheta n+2) + \cdots$$
$$+\phi (\vartheta n, \vartheta m) \phi (\vartheta n+1, \vartheta m) \phi (\vartheta n+2, \vartheta m) \cdots \phi(\vartheta m-2, \vartheta m)b\phi(\vartheta m-2, \vartheta m)$$
$$+\phi(\vartheta n, \vartheta m)\phi(\vartheta n+1, \vartheta m)\phi(\vartheta n+2, \vartheta m) \cdots \phi(\vartheta m-2, \vartheta m)b\phi(\vartheta m-1, \vartheta m).$$

Since $\phi \geq 1$, we conclude that,

$$\Phi (\vartheta n, \vartheta m) \phi (\vartheta n+1, \vartheta m) \phi (\vartheta n+2, \vartheta m) \cdots \phi (\vartheta m-2, \vartheta m) b\phi (\vartheta m-1, \vartheta m)$$
$$\leq \phi(\vartheta n, \vartheta m)\phi(\vartheta n+1, \vartheta m)\phi(\vartheta n+2, \vartheta m) \cdots \phi(\vartheta m-2, \vartheta m)\phi(\vartheta m-1, \vartheta m)b\phi(\vartheta m-1, \vartheta m),$$

And hence,

$$b\phi(\vartheta n, \vartheta m) \leq \phi(\vartheta n, \vartheta m) b\phi(\vartheta n, \vartheta n+1) + \phi(\vartheta n, \vartheta m)\phi(\vartheta n+1, \vartheta m) b\phi(\vartheta n+1, \vartheta n+2) + \cdots + \phi(\vartheta n, \vartheta m)\, \phi(\vartheta n+1, \vartheta m)\, \phi(\vartheta n+2, \vartheta m) \cdots \phi(\vartheta m-1, \vartheta m) b\phi(\vartheta m-1, \vartheta m,$$

Which completes the proof.
Now, we state and prove our first main theorem.

Theorem 1.2. Let (X, bϕ,⪯) be an ordered extended b-metric space having the t-property. Let f: X → X be a monotone non-decreasing self-map such that,
(C1) there exists ϑ0 ∈ X with ϑ0 ⪯ fϑ0,
(C2) for all ϑ, y ∈ X with ϑ ≺ y,

$$d(y, f y) \leq \lambda d(\vartheta, f\vartheta), \text{ where } \lambda \in (0, 1) \quad (1)$$

Further, suppose that the mapping ϕ: X × X → [1, ∞) satisfies

$$\phi(\vartheta, z) \geq \phi(y, z), \quad (2)$$

for all ϑ, y ∈ X with ϑ ≺ y and anyz ∈ X. Suppose also that limm,n→∞
ϕ(ϑm, ϑn) <1/λ, where ϑn = f^n ϑ0, n ∈ N. Then f has a fixed point in X.

Proof
By the assumption (C1), there exists ϑ0 ∈ X with ϑ0 ⪯ fϑ0. Starting with this element ϑ0, define the sequence {ϑn}, n ∈ N0 as ϑn+1 = fϑn. If ϑn+1 = ϑn for some n ≥ 0, the proof is done. Assume that ϑn+1,
ϑn for all n ≥ 0. From the assumption (C1) we have ϑ0 ≺ ϑ1, and using the fact f is non-decreasing, we deduce ϑ2 = fϑ1 ≺ fϑ0 = ϑ1. By continuing the process, we conclude that the sequence {ϑn} is strictly increasing. Now, since ϑ0, ϑ1 ∈ X with ϑ0 ≺ ϑ1, then by (1), we have

$$b\phi(\vartheta 1, f\vartheta 1) \leq \lambda b\phi(\vartheta 0, f\vartheta 0). \quad (3)$$

Again, since ϑ1, ϑ2 ∈ X with ϑ1 ≺ ϑ2, then by (1), we have

$$b\phi(\vartheta 2, f\vartheta 2) \leq \lambda b\phi(\vartheta 1, f\vartheta 1) \quad (4)$$

Using (3) in (4), we get

$$b\phi(\vartheta 2, f\vartheta 2) \leq \lambda^2\, b\phi(\vartheta 0, f\vartheta 0).$$

Continuing in this way, we get

$$b\phi(\vartheta n, f\vartheta n) \leq \lambda^n b\phi(\vartheta 0, f\vartheta 0), \, n \in N \quad (5)$$

Now we will show that {ϑn} is a Cauchy sequence. Let n, m ∈ N with n < m. Then by Lemma 3.1, we have

$$b\phi(\vartheta n, \vartheta m) \leq \phi(\vartheta n, \vartheta m)\, b\phi(\vartheta n, \vartheta n+1) + \phi(\vartheta n, \vartheta m)\phi(\vartheta n+1, \vartheta m). b\phi(\vartheta n+1, \vartheta n+2) + \cdots + \phi(\vartheta n, \vartheta m)\phi(\vartheta n+1, \vartheta m)\phi(\vartheta n+2, \vartheta m) \cdots \phi(\vartheta m-1, \vartheta m) b\phi(\vartheta m-1, \vartheta m). \quad (6)$$

Since {ϑn} is strictly increasing sequence, then by using the property (2) of ϕ, we have

$$\phi(\vartheta m-1, \vartheta m) \leq \phi(\vartheta m-2, \vartheta m) \leq \ldots \leq \phi(\vartheta n+1, \vartheta m) \leq \phi(\vartheta n, \vartheta m), \quad (7)$$

For all n, m ∈ N with n < m. Now by using (7) in (6), we get

$$b\phi(\vartheta n, \vartheta m) \leq \phi(\vartheta n, \vartheta m) b\phi(\vartheta n, \vartheta n{+}1) + [\phi(\vartheta n, \vartheta m)]^2 \, b\phi(\vartheta n{+}1, \vartheta n{+}2) + [\phi(\vartheta n, \vartheta m)]^3 b\phi(\vartheta n{+}2, \vartheta n{+}3) + \cdots + [\phi(\vartheta n, \vartheta m)]^{m-n-1} b\phi(\vartheta m{-}1, \vartheta m)$$

By using (5) in (8), we have

$$\begin{aligned} b\phi(\vartheta n, \vartheta m) &\leq [\phi(\vartheta n, \vartheta m)] b\phi(\vartheta n, \vartheta n{+}1) + [\phi(\vartheta n, \vartheta m)]^2 b\phi(\vartheta n{+}1, \vartheta n{+}2) + \cdots + [\phi(\vartheta n, \vartheta m)]^{m-n-1} b\phi(\vartheta m{-}1, \vartheta m), \\ &\leq [\phi(\vartheta n, \vartheta m)] \lambda^n \, b\phi(\vartheta 0, f\vartheta 0) + [\phi(\vartheta n, \vartheta m)]^2 \lambda^{n+1} \, b\phi(\vartheta 0, f\vartheta 0) + \cdots + [\phi(\vartheta n, \vartheta m)]^{m-n-1} \lambda^{m-1} b\phi(\vartheta 0, f\vartheta 0), \\ &\leq [\phi(\vartheta n, \vartheta m)\lambda]^n + [\phi(\vartheta n, \vartheta m)\lambda]^{n+1} + \cdots + [\phi(\vartheta n, \vartheta m)\lambda]^{m-1}] b\phi(\vartheta 0, f\vartheta 0) \\ &= [t^n + t^{n+1} + \cdots + t^{m-1}] \, b\phi(\vartheta 0, f\vartheta 0) \\ &= [t^n (1 - t^{m-n-1})/1 - t] b\phi(\vartheta 0, f\vartheta 0), \end{aligned}$$

Where $t = \phi(\vartheta n, \vartheta m)\lambda$. Using the fact that $\lim_{n,m\to\infty} \phi(\vartheta n, \vartheta m) < 1/\lambda$, we have $\lim_{n,m\to\infty} t < 1$. Hence, passing to limit as $n, m \to \infty$ in the inequality above, we conclude that,

$$\lim_{n,m\to\infty} b\phi(\vartheta n, \vartheta m) \leq \lim_{n,m\to\infty} [t^n (1 - t^{m-n-1})/1 - t] b\phi(\vartheta 0, f\vartheta 0) = 0 \quad (9)$$

This proves that $\{\vartheta n\}$ is a strictly increasing Cauchy sequence and since $(X, b\phi, \preceq)$ has the t-property, there is a $\omega \in X$, such that $\vartheta n \prec \omega$, $\forall n \in N$. Thus, by using (1) and (5), we have

$$\begin{aligned} b\phi(w, f w) &\leq \lambda b\phi(\vartheta n, f\vartheta n), \\ &\leq \lambda^{n+1} b\phi(\vartheta 0, f\vartheta 0), \\ &\to 0, \text{ as } n \to \infty, \end{aligned}$$

Implying that $b\phi(w, f w) = 0$ and hence, f has a fixed point in X which completes the proof.
We provide some illustrations to support our theoretical result.
Example 1.3. Let $X = \{-1/2, -1/4, -1/8, ...\} \cup \{0\}$ and "$\preceq$" is defined as natural ordering "$\leq$". We define the metric $b\phi: X \times X \to [0, \infty)$ by

$$b\phi(\vartheta, \eta) = \begin{cases} 0, & \text{iff } \vartheta = \eta \\ 3 + \vartheta + \eta, & \text{iff } \vartheta \neq \eta. \end{cases}$$

Further, if we specify $\phi: X \times X \to [1, \infty)$ by $\phi(\vartheta, \eta) = 5 + \eta - \vartheta$, then we can easily verify that $(X, b\phi, \preceq)$ is an ordered extended b-metric space.
Now, we consider $f: X \to X$ by $f\vartheta = \vartheta/2$ and $\lambda = 1/7$.
Here, for any $\vartheta 0 \in X$, we can show that $\vartheta k = f^k(\vartheta 0) = -1/2^k$ for some $k \in N \cup \{0\}$. Thus, $\phi(\vartheta m, \vartheta n) = 5 - 1/2^n + 1/2^m$ and hence $\lim_{m,n\to\infty} \phi(\vartheta m, \vartheta n) = 5 < 7 = 1/\lambda$.
Now, it remains to prove (1). Let $\vartheta, \eta \in X$ with $\vartheta < \eta$, we have

$$\begin{aligned} \lambda b\phi(\vartheta, f\vartheta) - b\phi(\eta, f\eta) &= 1/7[b\phi(\vartheta, \vartheta/2)] - b\phi(\eta, \eta/2) \\ &= 1/7[3 + \vartheta + \vartheta/2] - [3 + \eta + \eta/2] \\ &\geq 0. \end{aligned}$$

Thus, all of the requirements of Theorem 1.2 have been met. Hence, f has a fixed point in X which is 0.

1.2. Fixed points in O-complete ordered extended b-metric spaces

We will first review some definitions before moving on to the major finding.

Definition 1.4. [2]. An ordered metric space $(X, b\phi,\preceq)$ is said to be O-complete if every increasing Cauchy sequence in X converges in X. In an ordered metric space, completeness implies O-completeness, but the converse is not true in general.

Example 1.5. Let $X = (0, \infty)$ induced with the natural ordering and $b\phi(\vartheta, y) = |\vartheta - y|$, then clearly $(X, b\phi,\preceq)$ is O-complete but not complete.

Theorem 1.6. Let $(X, b\phi,\preceq)$ be an O-complete ordered extended b-metric space. Let $f: X \to X$ be a self-mapping which is continuous, monotone non-decreasing and satisfies,

(D1) there exists $\vartheta 0 \in X$ such that $\vartheta 0 \preceq f\vartheta 0$.

(D2) for all $\vartheta, y \in X$ with $\vartheta \prec y, \vartheta, f(\vartheta)$, the inequality

$$b\phi(y, f y) \leq \lambda[b\phi(\vartheta, y) + b\phi(f\vartheta, f y] \quad (10)$$

Holds for some $\lambda \in (0,1/2)$. Further, suppose that the mapping $\phi: X \times X \to [1, \infty)$ is such that

$$\phi(\vartheta, z) \geq \phi(y, z) \quad (11)$$

Holds for all $\vartheta, y \in X$ with $\vartheta \prec y$ and for any $z \in X$. Let also limm, $n\to\infty$

$$\phi(\vartheta m, \vartheta n) <(1 - \lambda)/\lambda, \text{ where } \vartheta n = f^\wedge n\vartheta 0, n \in N.$$

Then f has a fixed point in X.

Proof. As in Theorem 1.2, starting with $\vartheta 0 \in X$ in the condition (D1), we construct a strictly increasing sequence $\{\vartheta n\}$ in X as $\vartheta n+1 = f\vartheta n$, $n \in N$.

Since $\vartheta 0 \prec \vartheta 1$, we replace y, ϑ in (10) by $\vartheta 1, \vartheta 0$ respectively, and we get

$$B\phi (\vartheta 1, f\vartheta 1) \leq \lambda [b\phi (\vartheta 0, \vartheta 1) + b\phi (f\vartheta 0, f\vartheta 1]$$
$$= \lambda b\phi (\vartheta 0, f\vartheta 0) + \lambda b\phi (\vartheta 1, f\vartheta 1).$$

Then

$$b\phi(\vartheta 1, f\vartheta 1) \leq[\lambda/1 - \lambda]b\phi(\vartheta 0, f\vartheta 0) \quad (13)$$

Again as $\vartheta 1 \prec \vartheta 2$, by using (10) with $y = \vartheta 2$, $\vartheta = \vartheta 1$ and (12), we obtain

$$b\phi(\vartheta 2, f\vartheta 2) \leq \lambda[b\phi(\vartheta 1, \vartheta 2) + b\phi(f\vartheta 1, f\vartheta 2],$$
$$= \lambda b\phi(\vartheta 1, f\vartheta 1) + \lambda b\phi(\vartheta 2, f\vartheta 2).$$

Then,

$$b\phi(\vartheta 2, f\vartheta 2) \leq (\lambda/1 - \lambda)b\phi(\vartheta 1, f\vartheta 1),$$

Which implies,

$$b\phi(\vartheta 2, f\vartheta 2) \leq(\lambda/1 - \lambda)^\wedge 2b\phi(\vartheta 0, f\vartheta 0),$$

Upon using (13). Continuing this process, we get

$$B\phi (\vartheta n, f\vartheta n) \leq (\lambda/1 - \lambda)^\wedge n\, b\phi(\vartheta 0, f\vartheta 0),$$

For all $n \in N$. Since $0 < \lambda < 1/2$, then $0 < k =\lambda/1-\lambda< 1$. Now, the inequality (14) becomes

$$b\phi (\vartheta n, f (\vartheta n)) \leq k^\wedge nb\phi(\vartheta 0, f\vartheta 0) \quad (15)$$

As in the proof of Theorem 3.2, we can show that $\{\vartheta n\}$ is an increasing Cauchy sequence in X. Since $(X, b\phi,\preceq)$ is O-complete, there exists $w \in X$ such that

$$\text{limn}\to\infty\vartheta n = \omega \quad (16)$$

Since f is continuous, we conclude that

$$\omega = \lim_{n\to\infty} \vartheta_{n+1} = \lim_{n\to\infty} f(\vartheta_n) = f(\omega) \quad (17)$$

Hence w is a fixed point off in X which ends the proof.

1.3. Fixed points of extended b-metric space in the sense of Boyd-Wong

In this section, we state and prove our last main result which is a fixed point theorem for contractions of Boyd-Wong type on ordered, extended b-metric spaces with the t-property. The Boyd-Wong contractions [16] are known to be one important extension of the Banach contractions and studied by many authors [13, 22].

First, we recall the auxiliary functions involved in the definition of Boyd-Wong contractions.

Let Ψ be set of all functions $\psi: [0, \infty) \to [0, \infty)$ satisfying,

(i) ψ is non-decreasing,

(ii) $\psi(x) < x$, $\forall\, x > 0$,

(iii) $\lim_{r\to x^+}\psi(r) < x$, $\forall\, x > 0$.

We will use the following lemma, the proof of which can be found in [8].

Lemma 1.7. [8]. Let $\psi \in \Psi$ and $\{u_n\}$ be a given sequence such that $u_n \to 0^+$ as $n \to \infty$. Then $\psi(u_n) \to 0^+$ as $n \to \infty$. Also $\psi(0) = 0$.

Theorem 3.8. Let $(X, b_\phi, \preceq)$ be an ordered extended b-metric space having the t-property and $f: X \to X$ be a monotone non-decreasing self-mapping. Assume that for all $\vartheta, y \in X$ with $\vartheta \prec y$, we have

$$b_\phi(y, f y) \le \psi(b_\phi(\vartheta, f\vartheta)), \quad (18)$$

Where $\psi \in \Psi$. Suppose that the series $\Sigma\psi^n(t)$ converges for all $t > 0$ and there exists $\vartheta_0 \in X$ such that $\vartheta_0 \preceq f(\vartheta_0)$.
Suppose also that the mapping $\phi: X \times X \to [1, \infty)$ satisfies for all $\vartheta, y \in X$ with $\vartheta \prec y$

$$\phi(\vartheta, z) \ge \phi(y, z),$$

For all $\vartheta, y \in X$ with $\vartheta \prec y$ and any $z \in X$, and $\lim_{m,n\to\infty}\phi^m(\vartheta_m, \vartheta_n) = L$,

Where $L < \infty$, and $\vartheta_n = f^n\vartheta_0$, $n \in N$. Then f has a fixed point in X. Moreover, every strict upper bound of fixed point off is also a fixed point off.

Proof. The proof starts as the proof of Theorem 3.2 by constructing a strictly increasing sequence $\{\vartheta_n\}$ in X defined by

$$\vartheta_{n+1} = f\vartheta_n. \quad (19)$$

Denote $T_n = b_\phi(\vartheta_n, f\vartheta_n)$, for all $n \in N_0$. Since $\vartheta_n, f\vartheta_n$, we have $T_n > 0$ for all $n \in N_0$. Also, using the fact that $\vartheta_n \prec \vartheta_{n+1}$ for all $n \in N$, from (18), we have

$$T_{n+1} = b_\phi(\vartheta_{n+1}, f\vartheta_{n+1}) \le \psi(b_\phi(\vartheta_n, f\vartheta_n)) = \psi(T_n) < T_n. \quad (20)$$

This shows that $\{T_n\}$ is a monotone decreasing sequence in R^+ so, there exists $r \ge 0$ such that

$$\lim_{n\to\infty} T_n = r. \quad (21)$$

Letting $n \to \infty$ in (20), we get

$$r \le \lim_{n\to\infty} \psi(T_n) < r,$$

which implies

$$\lim_{n\to\infty} \psi(T_n) = r. \quad (22)$$

Suppose that $r > 0$. By (22) and the property (iii) of the function ψ, we get

$$r = \lim_{n\to\infty} \psi(T_n) = \lim_{T_n\to r^+} \Psi(T_n) < r,$$

Which is a contradiction, so that, r = 0, and

limn→∞

Tn = 0. (23)

Now, the condition (18) with y = ϑ1, ϑ = ϑ0, we have

bϕ (ϑ1, fϑ1) ≤ ψ(bϕ(ϑ0, fϑ0).

Repeating this process n times, we deduce

Tn = bϕ (ϑn, fϑn) ≤ ψ

n

(bϕ (ϑ0, fϑ0)), for all n ≥ 1.

Since P

n≥1 ψ

n

(t) converges for all t > 0, we have that P

n≥1 Tn converges.

We shall show that {ϑn} is a Cauchy sequence in X. As {ϑn} is a strictly increasing sequence, for n, m ∈ N
With n < m, by using the triangle inequality, (18), (19), (23) and the definition of ϕ, we obtain

bϕ (ϑn, ϑm) ≤ ϕ (ϑn, ϑm)bϕ(ϑn, ϑn+1) + [ϕ(ϑn, ϑm)]2

bϕ (ϑn+1, ϑn+2)

+ [ϕ (ϑn, ϑm)] 3

bϕ (ϑn+2, ϑn+3) + · · · + [ϕ(ϑn, ϑm)]m−n−1

bϕ (ϑm−1, ϑm)

≤ [ϕ (ϑn, ϑm)] m

[Tn + Tn+1 + · · · + Tm−1]

≤ [ϕ (ϑn, ϑm)] mX∞

k=n

Tk

, (24)

Due to the fact that lim n, m→∞

ϕ(ϑn, ϑm)]m

Is finite and the series P
n≥1 Tn is convergent, its tail P∞

kn

Tk = 0 → 0 as

n → ∞ and we have,

lim n,m→∞

ϕ(ϑn, ϑm)]mX∞

kn

Tk = 0,

Which implies that

$\lim_{n, m\to\infty} b\phi(\vartheta n, \vartheta m) = 0.$

Hence, $\{\vartheta n\}$ is a monotone increasing Cauchy sequence in X, which has the t-property, so there exists $w \in X$ such that $\vartheta n \prec w$ for all n. By using (18) and (22), we have

$$B\phi (w, f w) \leq \psi (b\phi (\vartheta n, f\vartheta n)) = \psi (Tn) \to 0, \text{ as } n \to \infty.$$

This shows that w is a fixed point off in X. Let $z \in X$ be any strict upper bound of w, i.e., $w \prec z$. By using (18) and Lemma 3.7, we have

$b\phi (z, f z) \leq \psi (b\phi (w, f w)) = \psi(0) = 0.$

Hence z is also a fixed point of in X

Conclusion

The study of contractive mappings in controlled metric spaces and extended b-metric spaces reveals significant insights into the nature of convergence and fixed points. These generalized spaces provide a framework that extends traditional metric space theory, allowing for the exploration of more complex relationships between points and their distances. The existence of fixed points under contractive mappings in these spaces underscores the robustness of this mathematical concept across various contexts. Additionally, the flexibility offered by controlled metrics and extended b-metrics opens new avenues for applications in areas such as functional analysis and nonlinear analysis. Future research can further investigate the implications of these mappings, potentially leading to novel solutions in both theoretical and applied mathematics.

Competing Interests

The author claims to have no interests that would be in conflict.

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