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Common Fixed Point Theorems and Non-Expansive Mapping in Banach Space

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Abstract
In this paper, we describe non-expansive mapping in Banach space and several popular fixed point theorems. Our goal is to apply the non-expansive mapping and theorems to Banach space.

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Introduction

We generalize a well-known Gregus (1980) [10] result by establishing a common fixed point theorem for self mappings that are not always commuting of a closed and convex subset of a Banach space.
For every α, β in X, let G be a mapping of X into itself that satisfies the inequality $\| G\alpha - G\beta \| \leq \| \alpha - \beta \|$. The class of contraction mapping is generally known to be non-expensive, and G is appropriately included in the class of all continuous mappings. For non-expansive mappings defined on a closed, bounded, and convex subset of a uniformly convex Banach space and in spaces with richer structure, Kirk (1965) [2] separately demonstrated a fixed point theorem.
Many authors have considered various generalizations of non-expansive mappings. Particularly noteworthy are the works of Goebel (1969) [6]; Goebel and Zlotkiewicz (1971) [7]; Goebel, Kirk, and Shimi (1973) [8]; Massa and Roux (1978) [9]; Dotson (1972a and b) [2, 3]; Emmanuele (1981) [5]; and Rhoades (1982) [11]. Kirk (1965, 1981, 1983) [2, 12, 13] provides a thorough overview of fixed point theorems for non-expansive and related mappings.
However, certain mappings have a unique fixed point and meet constraints that are comparable to those of non-expansive mappings. However, these mappings cannot be thought of as extensions of non-expansive mappings. Recent instances of this type can be found in Rhoades (1978) and Gregus (1980) [10]. Inspired by a contractive condition of Hardy and Rogers (1973) [14], we expand Gregus's (1980) [10] solution to the situation of two mappings in this chapter.
Let M be a subset of X that is closed and convex. In conclusion, this author demonstrated the following outcome under the assumption that $y = z$ in Gregus's (1980) [10] contractive condition.

Preliminaries

a) Banach space

A complete vector space with a norm is called a Banach space. If two norms provide the same topology, which is equivalent to the presence of constants and such that and hold for all, they are said to be comparable. All norms are comparable in the situation of finite dimensions. There are numerous possible standards for an infinite-dimensional space.

b) Non expansive mapping

Let C be a nonempty convex subset of a real Banach space E and $\mathbb{R}$ be the set of real numbers. A mapping $T: C \to C$ is called non expansive if $\| Tx - Ty \| \leq \| x - y \|$ for all $x, y \in C$.

Theorem 1:
Let G be a mapping of M by itself, so resolving the inequality

$$\| G\alpha - G\beta \| \leq x.\| \alpha - \beta \| + y.\{\| G\alpha - \alpha \| + \| G\beta - \beta \|\} \tag{1}$$

for all α, β in M, where $0 < x < 1, y > 0$ and $x + 2y = 1$. Then G has a unique fixed point.
The following theorem is now proven.

Theorem 2:
Let F and G be mappings of M into themselves satisfying the inequality

$$\begin{aligned} \| F\alpha - G\beta \| &\leq x.\| \alpha - \beta \| + y.\{\| F\alpha - \alpha \| + \| G\beta - \beta \|\} \\ &+ z.\{\| F\alpha - \beta \| + \| G\beta - \alpha \|\} \end{aligned} \tag{2}$$

for all α, β in M, where $0 < x < 1,\ y > 0$ and $x + 2y + 2z = 1$ and $(1 - y).z < xy$. If

$$\| G\alpha - \alpha \| \leq \| F\alpha - \alpha \| \tag{3}$$

$\forall\ \alpha$ in M, then F and G have a unique common fixed point w in M. Moreover, w is the unique fixed point of F and G.

Proof:
Let α be an arbitrary point in M. From (2), we deduce that

$$\begin{aligned} \| FG\alpha - G\alpha \| \leq x.\| G\alpha - \alpha \| &+ y.\{\| FG\alpha - G\alpha \| + \| G\alpha - \alpha \|\} \\ &+ z.\{\| FG\alpha - G\alpha \| + \| G\alpha - \alpha \|\} \end{aligned}$$

which implies that

$$\| FG\alpha - G\alpha \| \leq \frac{x+y+z}{1-y-z}.\| G\alpha - \alpha \| = \| G\alpha - \alpha \| \tag{4}$$

Similarly, we have

$$\| GF\alpha - F\alpha \| \leq \| F\alpha - \alpha \|. \tag{5}$$

Since (4) holds $\forall\ \alpha$ in M, we deduce that

$$\| FGF\alpha - FG\alpha \| \leq \| GF\alpha - F\alpha \|,$$

Which implies, by (3) and (5), that

$$\| GGF\alpha - GF\alpha \| \leq \| FGF\alpha - GF\alpha \| \leq \| F\alpha - \alpha \|. \tag{6}$$

We now define the point γ by

$$\gamma = \frac{1}{2}GF\alpha + \frac{1}{2}GGF\alpha$$

Then, it follows, from (6), that

$$2\| GF\alpha - \gamma \| = 2\| GGF\alpha - \gamma \| = \| GGF\alpha - GF\alpha \| \leq \| F\alpha - \alpha \|. \tag{7}$$

Since M is convex, γ belongs to M and using (2), (5), (6) and (7), we have that

$$\begin{aligned} 2\| F\gamma - \gamma \| &= \| 2F\gamma - (GF\alpha + GGF\alpha) \| = \| F\gamma - GF\alpha \| + \| F\gamma - GGF\alpha \| \\ &\leq \| F\gamma - GF\alpha \| + \| F\gamma - GGF\alpha \| \\ &\leq x.\| \gamma - F\alpha \| + y.\{\| F\gamma - \gamma \| + \| F\alpha - \alpha \|\} \\ &+ z.\{\| F\gamma - \gamma \| + \| F\alpha - \gamma \| + \| GF\alpha - \gamma \|\} \\ &+ x.\ \| \gamma - GF\alpha \| + y.\{\| F\gamma - \gamma \| + \| F\alpha - \alpha \|\} \\ &+ z.\{\| F\gamma - \gamma \| + \| GF\alpha - \gamma \| + \| GGF\alpha - \gamma \|\} \\ &\leq x.\{\| F\alpha - \gamma \| + \frac{1}{2}.\ \| F\alpha - \alpha \|\} + 2y.\{\| F\gamma - \gamma \| + \| F\alpha - \alpha \|\} \end{aligned} \tag{8}$$

$$+ z.\ \{2 \,||\, F\gamma - \gamma \,||\, + \,||\, F\alpha - \gamma \,||\, + \frac{3}{2} .\ ||\, F\alpha - \alpha \,||\}$$

On the other hand, using (2), (5) and (6), we obtain that

$$2 \,||\, F\alpha - \gamma \,||\, = \,||\, 2F\alpha - (GF\alpha + GGF\alpha) \,||\, = \,||\, F\alpha - GF\alpha \,||\, + \,||\, F\alpha - GGF\alpha \,|| \quad (9)$$
$$\leq \,||\, F\alpha - GF\alpha \,||\, + \,||\, F\alpha - GGF\alpha \,||$$
$$\leq \,||\, F\alpha - \alpha \,||\, + x .\, ||\, \alpha - GF\alpha \,||\, + y .\ \{\, ||\, F\alpha - \alpha \,||\, + \,||\, F\alpha - \alpha \,||\, \}$$
$$+ z .\ \{\, ||\, F\alpha - \alpha \,||\, + \,||\, GGF\alpha - GF\alpha \,||\, + \,||\, GF\alpha - F\alpha \,||\, + \,||\, F\alpha - \alpha \,||\, \}$$
$$\leq \,||\, F\alpha - \alpha \,||\, + x .\{\, ||\, F\alpha - \alpha \,||\, + \,||\, GF\alpha - F\alpha \,||\, \} + (2y + 4z) .\ ||\, F\alpha - \alpha \,||$$
$$\leq (1 + 2x + 2y + 4z) .\ ||\, F\alpha - \alpha \,||$$
$$= (3 - 2y) .\ ||\, F\alpha - \alpha \,||.$$

It is easily seen that (8) and (9) imply that

$$2 \,||\, F\gamma - \gamma \,|| \leq x .(2 - y) .\ ||\, F\alpha - \alpha \,||\, + 2y .\{||\, F\alpha - \alpha \,||\, + \,||\, F\gamma - \gamma \,||\}$$
$$+ z .\ \{2 \,||\, F\gamma - \gamma \,||\, + (3 - y) .||\, F\alpha - \alpha \,||\}.$$

Consequently, we have that

$$||\, F\gamma - \gamma \,|| \leq \delta .||\, F\alpha - \alpha \,||, \quad (10)$$

Where

$$\delta = \frac{1}{2} \left(\frac{2x - xy + 2y + 3z - yz}{1 - y - z}\right)$$

It follows that $0 < \delta < 1$ based on the assumptions made about the constants $x, y, and\ z$. We claim that $h = inf\{||\, F\alpha - \alpha \,|| : \alpha \in M\} = 0$, otherwise for $0 < \varepsilon < (1 - \delta). h/\delta$, there exists a point $\bar{\alpha}$ in M such that $||\, F\bar{\alpha} - \bar{\alpha} \,|| \leq h + \epsilon$ and hence (10) implies that $h \leq ||\, F\gamma - \gamma \,|| \leq \delta .||\, F\bar{\alpha} - \bar{\alpha} \,|| \leq \delta .(h + \epsilon) < h$, a contradiction.
Thus $h = 0$ and the sets

$$\mathbb{H}_n = \{\alpha \in M : \{||\, F\alpha - \alpha \,|| < \frac{1}{n}\}\}$$

are non-empty for any $n = 1, 2, \dots ;$

Now we have,

$$\mathbb{H}_1 \supseteq \mathbb{H}_2 \supseteq \dots\dots \supseteq \mathbb{H}_n \supseteq \dots\dots \quad (11)$$

Let $\overline{\mathbb{H}}_n$ be the closure of $\mathbb{H}_n$. We now show that

$$\text{diam}\ \overline{\mathbb{H}}_n \leq \frac{(3 - x)}{2yn} \quad (12)$$

for any $n = 1, 2, \dots \dots$ Indeed, we obtain on using (2) for all α, β in $\mathbb{H}_n$,

$$||\, \alpha - \beta \,|| \leq ||\, F\alpha - \alpha \,||\, + \,||\, F\alpha - \beta \,||$$
$$\leq ||\, F\alpha - \alpha \,||\, + \,||\, G\beta - \beta \,||\, + \,||\, F\alpha - G\beta \,||$$
$$\leq \frac{2}{n} + x .||\, \alpha - \beta \,||\, + y .\ \{||\, F\alpha - \alpha \,||\, + \,||\, G\beta - \beta \,||\} +$$
$$z .\{||\, F\alpha - \alpha \,||\, + \,||\, \alpha - \beta \,||\, + \,||\, G\beta - \beta \,||\, + \,||\, \alpha - \beta \,||\}$$
$$\leq \frac{2}{n} + (x + 2z) .||\, \alpha - \beta \,||\, + \frac{(2y + 2z)}{n}$$
$$= \frac{(3 - x)}{n} + (1 - 2y) .||\, \alpha - \beta \,||$$

By equation (3) $||\, G\beta - \beta \,|| \leq ||\, F\beta - \beta \,|| \leq \frac{1}{n}$. By above inequality (12)
$diam\ \mathbb{H}_n = diam\ \overline{\mathbb{H}}_n$ and clearly it follows from (11) that

$$\overline{\mathbb{H}}_1 \supseteq \overline{\mathbb{H}}_2 \supseteq \dots\dots \supseteq \overline{\mathbb{H}}_n \supseteq \dots\dots$$

The series $diam\ \overline{\mathbb{H}}_n$ converges to zero as $n \to \infty$ by (12), indicating that $\{\overline{\mathbb{H}}_n\}$ is a decreasing sequence of non-empty subsets of M. Cantor's intersection theorem states that since X and M are complete, there is a point w in M such that

$$w \in \bigcap_{n=1}^{\infty} \overline{\mathbb{H}}_n.$$

Accordingly, $|| \ Fw - w \ || \leq \frac{1}{n}$ for any $n = 1, 2, \ldots \ldots$, and so $Fw = w$, By using (3), we have $Gw = w$. Then, w is a fixed point that both F and G share. Assume that w' is an additional fixed point of F. With (2) applied to $\alpha = w \ and \ \beta = w'$ we obtain that

$$|| \ w' - w \ || = || \ Fw' - Gw \ ||$$

$$\leq x \,. || \ w' - w \ || + z \,. \{|| \ w' - w \ || + || \ w - w' \ ||\}$$

$$= (x + 2z) \,. || \ w' - w \ ||.$$

This implies that $w' = w$ since $x + 2z = 1 - 2y < 1$. Therefore w is the unique fixed point of F and similarly it is shown that w is the unique fixed point of G. This completes the proof.

Remark:
Theorem 2 becomes theorem 1 if $F = G$ and $z = 0$ are assumed.
The following outcome is obtained by enunciating theorem 2 for certain iterates of F and G.

Theorem 3:
The theorem States that the inequality is satisfied if F and G are mappings of M into themselves.

$$|| \ F_{u\alpha} - G_{v\beta} \ || \leq x \,. || \ \alpha - \beta \ || + y. \ \{|| \ F_{u\alpha} - \alpha \ || + || \ G_{v\beta} - \beta \ ||\}$$

$$+ z. \ \{|| \ F_{u\alpha} - \beta \ || + || \ G_{v\beta} - \alpha \ ||\}$$

for all α, β in M, where u and v are positive integers and x, y, z are as in theorem 2. If

$$|| \ G_{v\beta} - \alpha \ || \leq || \ F_{u\alpha} - \alpha \ ||$$

$\forall \ \alpha$ in M, then F and G have a unique common fixed point w in M. Further, w is the unique fixed point of F and G.

Proof:
By theorem 2, mapping F_u and G_v of M into itself have a unique common fixed point w in M. Since $Fw = FF_u w = F_u Fw$, we deduce that Fw is also a fixed point of F_u, it follows that $Fw = w$. Similarly, we can prove that $Gw = w$ and therefore w is common fixed point F and G. If w' is another fixed point of F, then we have that $F_u w' = w'$ but the uniqueness of w implies $w = w'$. Thus w is also the fixed point of F as well as for the mapping of G.
The following example shows the stronger generality of theorem 3 over theorem 2.

Example:
Let X be the Banach space of reals with Euclidean norm and $M = [0,2]$. We define F and G by putting $F\alpha = 0 \ if \ 0 \leq \alpha < 1$, $F\alpha = \frac{3}{5} \ if \ 1 \leq \alpha \leq 2$, $G\alpha = 0 \ if \ 0 \leq \alpha < 2 \ and \ G\alpha = \frac{9}{5}$ Then the condition (2) of theorem 1 does not hold,

Otherwise, we should have for $\alpha = 1$ and $\beta = 2 \frac{6}{5} = || \ F_1 - G_2 \ || \leq x. || \ 2 - 1 \ || + y. \{|| \ 1 - \frac{3}{5} \ || + || \ 2 - \frac{9}{5} \ ||\} + z. \{|| \ \frac{9}{5} - 1 \ || + || \ 2 - \frac{3}{5} \ ||\}$

$$= x + \frac{3y}{5} + \frac{11z}{5}$$

$$= 1 - 2y - 2z + \frac{3y}{5} + \frac{11z}{5}$$

Which implies $\frac{1}{5} + \frac{7y}{5} \leq \frac{z}{5}$, $i.e, 1 + 7y \leq z$, a contradiction. However, the conditions of theorem 3 are trivially satisfied for

$u = v = 2$ Since $F2\alpha = G2\alpha = 0$ for all α in M.
Although the contradictive condition used in this chapter is more general than (2), we explicitly note that the results for $F = G$ are not comparable to the results where the additional assumptions on the coefficients and the uniform convexity of X neither imply nor are implied by the assumptions of theorem 2.

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