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# Downward Axisymmetric Plumes and Some Possible Behaviours in a Quiescent Ambient

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## Abstract
Downward axisymmetric plumes with their possible behaviours have just been studied numerically, with the assumption that density is a quadratic function of temperature. Results here are for different values of $M_0$, where the value M0 = $\left(\frac{5}{2^5}\right)^{\frac{2}{5}}$ is termed critical forcing. Plumes with M0 value > $\left(\frac{5}{2^5}\right)^{\frac{2}{5}}$ are termed strongly forced plume. Within the interval $0 < M_0 < \left(\frac{5}{2^5}\right)^{\frac{2}{5}}$, two solutions are aimed: a warm weakly forced and cool weakly forced plume. Plume's uniqueness are in accordance with their respective $M_0$ values with the source offering some sort of forcing that is depended on the downward momentum. The warm weakly forced plume possess an unlimited temperature at its virtual source. At far
field where Q M0, M ~ Q in (38) and from equation (35), both volume and momentum flux advances linearly with distance, such that the dimensionless velocity V approach unity as Y → ∞. The results also showed that both warm weakly forced and critically forced plumes come to rest with final volume flux at $Q_f \leq \frac{1}{2}$ with an infinite width. The temperature here decreases monotonically with increasing initial momentum flux. Our plumes are assumed to have come from a virtual source condition, where giving interpretation as though they were from physical sources may be difficult. In terms of the study of plumes from physical sources, two types of such might be of practical importance: one of which is the power station warm discharge at a lake floor and a plume that descends from a surface gravity current. Those results in section 3.2 requires that it is important to prevent singularity at the virtual source (see Fig. 6) by making sure that the initial conditions suggest a physical source position at some point below the virtual source and this we have as a limitation. A more realistic model had been reached by George & Kay, 2017 for a fountain flow case, but need to be inverted in the case of a descending weakly forced plume. Lastly, with the entrainment model, it is expected that the plume will be fully turbulent. Thus, our results can be likened to the power station warm discharge which will definitely be fully turbulent with its large volume flux: these are basic conditions on the validity of these assumptions. The numerical results as presented here are very good as they present us with insight into downward plumes using (1).

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**Keywords:** Critically forced plume, Warm discharge, Momentum flux, Volume flux, Zero-buoyancy

## 1. Introduction
The disposal of cooling waste water from thermoelectric power plants are one of the main causes of some behaviours in our environment. It is known that this cooling waste water are mostly discharged at a temperature roughly $10^{\circ}C$ greater than the surrounding water from which it was taken and this in turn may affect the aquatic ecosystem negatively (George & Osaisai, 2023; George *et al*. 2023) [4, 5]. If we assume that the cooling water is drawn from either lake, pond or river then it is certain that the disposed waste water must be less dense than that of the aforementioned water sources (Macqueen, 1979; Kay, 2007; George & Kay, 2017) [13,8, 9]. The flow can be laminar or turbulent which is totally dependent on the discharge velocity. However, it is believed that a discharge from such a power station will definitely be turbulent with its large volume flux.
Now, if the discharged water is initiated at the floor of the medium or somewhere close to it and knowing that the discharged water is less dense, then it is certain that this lighter fluid will move upwards in the surrounding water into which the lighter

warm water was disposed forming a rising plume. The upward plume will entrain ambient fluid into itself even as it rises and this entrainment process will cause the plume to cool further as its temperature reduces gradually. If it happens that the temperature of the ambient water is below the temperature of maximum density which is about $4^{\circ}C$ for most numerical calculation, then it is believed that the rising fluid will generate a mixture denser than both the discharged and the ambient water. This newly generated mixed water will descend, sinking to the floor of the domain as the case may be. And on the floor, the descended dense water will again spread outwards as density current (Kay, 2007; George & Kay, 2017) [9]. A similar kind flow were also found in Lake Michigan by Hoglund and Spigarelli (1972) [10] as also recorded in the literature by George & Kay, 2017 [8]. Nevertheless, if the domain into consideration has a reduced depth then this upward flowing warm fluid could get to the surface and spread outwards as surface gravity current entraining more cold water itself (Kay, 2007; George & Kay, 2017) [9].

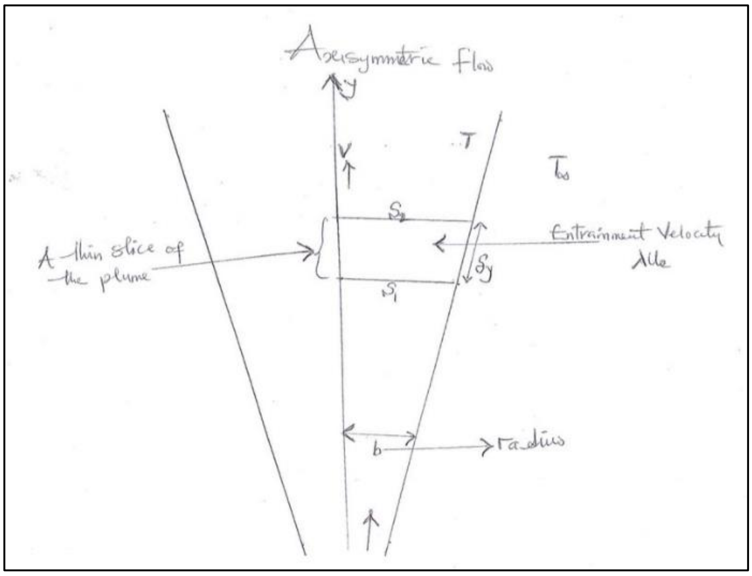


**Fig 1:** The sketch of a thin slice of the plume on which to perform mass, momentum and thermal energy budgets

Generally speaking, it is obvious that whenever water masses of different densities come in contact there will be different phases of flow: That is, It will be an upward flow (rising plume) if and only if the discharged is made at the floor or at some point in the surrounding water but not at the surface and the discharge water less dense than the surrounding water. Next will be a descending plume (i.e., an initially rising plume may later become a fountain) especially, any mixed fluid that have attained temperature of maximum density or any temperature close to it; or a surface gravity current at the surface level if the domain into consideration is not deep enough. Lastly is a density current along the floor of the domain as the case may be. Now, for the sake of this present stud our interest here is to consider only the descending stage of the plume and this is very important whenever water masses of different densities meet as also considered by Kay, 2007 for a symmetric flow cases. As such, the other flow cases are beyond the scope of our study and will be looked at in another investigation. We are to analysis the dynamical aspect of the descending dense water and assume a line plume descending from the head of a broad (spanwise) gravity current. This case involves annular descending plumes following axisymmetric spreading across the surface from above a point outfall: these may be modelled as two-dimensional if the plume's width is small compared to its radial distance from the outfall. The ambient of consideration is assumed quiescent, uniform and unstratified with a discharge temperature 10◦C on the floor of a lake with a uniform ambient water temperature 0◦C. Buoyancy reversal is one of the significant features that are associated with rising plumes. In fact, entrainment result to increase in density for rising plumes: as both discharge and ambient water mixes further by entrainment, density difference in the mixed fluid will increase, reducing the penetrating speed of the frontal head and this will continue until its temperature attains 4◦C or a temperature close to it and descend. Though, this is different from those plumes with linear mixing properties, where entrainment always reduces buoyancy. Plumes and jets with buoyancy reversal have received much attention in the past and in different situations: though, with the assumption that density is a linear function of temperature. But then, Caulfield and Woods (1995) [2]; Kay (2007) [12]; have also considered the plume entrainment equations with density as a quadratic function of mixing ratio. For those by Kay (2007) [12], entrainment always increases density as mixed fluid increases until its temperature had reached 4◦C and also show some similarities with those by Bloomfield & Kerr (2000) [1] and Osaisai & George (2024) [15] for axisymmetric fountains. Though, in the investigations by Caulfield and Woods (1995) [2], mixing between plume and ambient fluid always resulted to a decrease in density.

## 2. Model Formulation and Governing Equations with Scalings

We consider here a two-dimensional axisymmetric fountain progressing vertically from the bottom through a homoge- neous and quiescent water, as illustrated in Figure 1. Entrainment and dilution taking place mainly along its sides, the buoyancy and vertical velocity $v$ within the plume depend on both distance $y$ above the source and radial distance $b$ from the centreline. For a successful formulation, it is ideal to consider the entrainment of surrounding water by the plume and the non-monotonic dependence of density $\rho$ on temperature $T$ which are also key factors that describe a rising turbulent plume. A quadratic dependence relation of density on temperature is also assumed as consider by Kay, (2007) [9], George & Kay (2017) [8], George

& Kay (2022) [6] and George & Osaisai (2022) [6] so as to have a good illustration of the real dynamics.

$$\rho_m - \rho = \beta(T - T_m)^2 \quad (1)$$

Note that the terms in Equation (1) are the same as those used in Kay (2007) [12], George & Kay, (2022) [6] with the same explanation. The Morton *et al*. (1956) [14] hypothesis on entrainment is used that ambient fluid is entrained at a velocity proportional to the vertical velocity within the plume. All liquid properties are assumed constant except for the water density that changes with temperature and in turn gives rise to buoyancy force. Plumes that undergoes buoyancy reversal at the point where mixture attains temperature of maximum density is our main focus (Kay, 2007) [12].

It is a fact that the classical models for plumes uses equations for conservation of mass, momentum and buoyancy. Though, conservation of buoyancy is used when the buoyancy is a linear function of thermal energy, salinity, etc. (Kay, 2007) [12]. Whereas, buoyancy is taken as a nonlinear function of temperature and it is proportional to thermal energy. Thus, we have to derive our equations from the conservation laws for mass, momentum and thermal energy with the nonlinearity appearing in the buoyancy forcing term in the momentum equation similar to those by Kay (2007) [12]. Boussinesq approximation is also applied and we are using the top hat profiles, which are equivalent to replacing the mass, momentum and thermal fluxes by their mean values defined by integrating across the plume.

$$b^2 v = \int_0^\infty 2rv\,dr \quad (2)$$

$$b^2 v^2 = \int_0^\infty 2rv^2\,dr \quad (3)$$

$$b^2 vT = \int_0^\infty 2rvT\,dr \quad (4)$$

Surrounding water of homogeneous temperature $T\infty$ and density $\rho\infty$ is entrained into the plume at a velocity $ue$ assumed to be proportional to the vertical velocity (Morton *et al*., 1956) [14]:

$$u_e = \lambda|v| \quad (5)$$

The entrainment constant $\lambda$ have an acceptable value of about 0.08 for top-hat profiles (Turner, 1973) [16]. Though, the value $\lambda$ = 0.1 is used in most numerical calculation or scaled out as the case may be. By considering $|v|$ as a factor in our equations that describes entrainment together with other assumption, we stand to say that this investigation will provide us with important information on some parameters.

Considering the mass conservation budget over the thin slice of the plume as shown in Figure 1. We have that Entrainment Mass Flux = Mass Flux out flow of $S_2$ - Mass Flux in flow $S_1$

$$Mass\ Flux\ in\ flow\ S_1 = \rho\pi b^2 v \quad (6)$$

$$Mass\ Flux\ out\ flow\ of\ S_2 = (\rho + \delta\rho)\pi(b + \delta b)^2(v + \delta v) \quad (7)$$

$$Entrainment\ Mass\ Flux = 2\lambda\pi bv\rho_\infty\delta y \quad (8)$$

From equation (6) - (8), after using the Boussinesq approximation (that density variations will be ignored except in the buoyancy term) we have equation for volume flux as follows:

Entrainment volume Flux = volume Flux out flow of $S_2$ - volume Flux in flow $S_1$

$$2\lambda\pi bv\delta y = \pi(b + \delta b)^2(v + \delta v) - \pi b^2 v \quad (9)$$

After expanding further, and divide through by $\delta y$ and taking limit as $\delta b \to 0$ we have

$$\frac{d(b^2 v)}{dy} = 2\lambda bv \quad (10)$$

In like manner we determine the vertical momentum flux over the same thin slice. So that we can write

Upward buoyancy force = Momentum Flux out flow of $S2$ - Momentum Flux in flow $S1$. Note here that, we will make use of the transformed buoyancy term" $(\rho\infty - \rho) = \beta[(T - Tm) + (T\infty - Tm)](T - T\infty)$" derived from equation (1) above. Also, the upper

and lower signs refer to upward and downward direction plumes respectively (Kay, 2007).
Thus,

$$\frac{\mp\pi b^2 g\beta}{\rho_m}[(T-T_m)+(T_\infty-T_m)](T-T_\infty)\delta y=\pi(b+\delta b)^2(v+\delta v)^2-\pi b^2v^2 \quad (11)$$

After expanding further also, and divide through by $\delta y$ and taking limit as $\delta b \to 0$ and $(\delta v)^2 \to 0$ we have

$$\frac{d(b^2v^2)}{dy} = \mp\frac{b^2 g\beta}{2\rho_m}[2T_m-T-T_\infty](T-T_\infty) \quad (12)$$

Lastly, we determine the Thermal flux and we have that:
Entrainment Thermal Flux = Thermal Flux out flow of $S_2$ - Thermal Flux in flow $S_1$

$$2\lambda\pi bvT_\infty\delta y=\pi(b+\delta b)^2(v+\delta v)T-\pi b^2vT \quad (13)$$

$$2\lambda bvT_\infty\delta y=(b^2+2b\delta b+(\delta b)^2)(v+\delta v)T-b^2vT \quad (14)$$

After expanding further as usual, and divide through by $\delta y$ and taking limit as $\delta b \to 0$ we have

$$\frac{d(b^2vT)}{dy} = 2\lambda bvT_\infty \quad (15)$$

Hence, the governing Volume, Momentum and Thermal Flux are:

$$\frac{d(b^2v)}{dy} = 2\lambda bv \quad (16)$$

$$\frac{d(b^2v^2)}{dy} = \mp\frac{b^2 g\beta}{2\rho_m}[2T_m-T-T_\infty](T-T_\infty) \quad (17)$$

$$\frac{d(b^2vT)}{dy} = 2\lambda bvT_\infty \quad (18)$$

Combining equation (16) and (18) gives:

$$\frac{d(b^2vT)}{dy}=T_\infty\frac{d(b^2v)}{dy} \implies \frac{d}{dy}[b^2v(T-T_\infty)]=0 \quad (19)$$

Upon integration we have

$$b^2v(T-T_\infty)=F=Constant \quad (20)$$

$$T=T_\infty+\frac{F}{b^2v} \quad (21)$$

Equation (21) is the temperature in the plume and F is the relative thermal flux which is conserved because of the unstratified ambient condition. Using equation (21), we can rewrite (17) as follows:

$$\frac{d(b^2v^2)}{dy}=\mp\frac{g\beta}{2\rho_m}\frac{F}{v}[2T_m-2T_\infty-\frac{F}{b^2v}] \quad (22)$$

From equation (16) If we let

$$q=b^2v \quad (23)$$

And from equation (17) if we let

$$m=b^2v^2 \quad (24)$$

From (23) and (24) we can write

$$v = \frac{q}{b^2} \quad and \quad v = \frac{m^{\frac{1}{2}}}{b} \tag{25}$$

From equation (25) if we make b the subject we have:

$$b = \frac{q}{m^{\frac{1}{2}}} \tag{26}$$

which is the plume width. In like manner we can write for plume velocity from (23) and (24) as follows:

$$v = \frac{m}{q} \tag{27}$$

From here we can rewrite the volume and momentum flux as dependent variables rather than having the in terms of width and velocity. Then equation (16) and (22) becomes:

$$\frac{dq}{dy} = 2\lambda m^{\frac{1}{2}} \tag{28}$$

$$\frac{dm}{dy} = \mp \frac{g\beta}{2\rho_m} \frac{Fq}{m}[2(T_m - T_\infty) - \frac{F}{q}] \tag{29}$$

Now, we can consider the scaling parameters of this given problem where temperature scale we have to be ($T_m - T_\infty$), the conserved thermal flux F and the Buoyancy scale we have to be

$$g_m = \frac{g\beta}{\rho_m}(T_m - T_\infty)^2 \tag{30}$$

Also, we have the volume flux scale through the combination of both the conserved thermal flux and the temperature as follows:

$$q_T = \frac{F}{(T_m - T_\infty)} \tag{31}$$

Substituting (30) and (31) into (29) we have

$$\frac{dm}{dy} = \mp \frac{g\beta}{2\rho_m} \frac{q_T(T_m - T_\infty)q}{m}[2(T_m - T_\infty) - \frac{q_T(T_m - T_\infty)}{q}]$$

$$\frac{dm}{dy} = \mp \frac{q_T q}{2m}[2g_m - \frac{q_T g_m}{q}]$$

$$\frac{dm}{dy} = \mp \frac{q_T q g_m}{2m}[2 - \frac{q_T}{q}] \tag{32}$$

Our new set of equation becomes (28) and (32) we can define dimensionless variables for the problem as follows:

$$q = Qq_T, \quad y = (\frac{q_T^2}{\lambda^4 g_m})^{\frac{1}{5}}Y, \quad m = (\frac{q_T^6 g_m^2}{\lambda^2})^{\frac{1}{5}}M, \quad \phi = \frac{T - T_\infty}{(T_m - T_\infty)}, \quad b = (\frac{\lambda q_T^2}{g_m})^{\frac{1}{5}}B, \quad v = (\frac{q_T g_m^2}{\lambda^2})^{\frac{1}{5}}V \tag{33}$$

Where we are scaling out the entrainment coefficient λ so that our results are independent of its numerical value. The thermal flux equation (20) yields a relation between dimensionless temperature and volume flux as

$$\phi = \frac{1}{Q} \tag{34}$$

Using also (33) on (28) and (32) so as to have the equation of motion in dimensionless form as follows:

$$\frac{dQ}{dY} = 2M^{\frac{1}{2}} \tag{35}$$

$$\frac{dM}{dY} = \mp\frac{2Q-1}{2M} \tag{36}$$

Combining (35) and (36) to remove Y we have:

$$\frac{dM}{dQ} = \mp\frac{2Q-1}{4M^{\frac{3}{2}}} \tag{37}$$

Upon integration we have the solution to be

$$M^{\frac{5}{2}} = M_0^{\frac{5}{2}} \mp \frac{5}{8}(Q^2 - Q) \tag{38}$$

Where $M_0$ is the value of M at Q = 0. We can also determine the dimensionless variables for (26) and (27) which are the plume width and velocity respectively. For plume dimensionless width we have

$$B = \frac{Q}{M^{\frac{1}{2}}} \tag{39}$$

For plume dimensionless velocity we also have

$$V = \frac{M}{Q} \tag{40}$$

There is also need for us to relate M0 to conditions at the physical source. The plumes behaviour is governed by dimensionless temperature and Froude number at the source (Kay, 2007), defined as

$$\phi_s = \frac{T_s - T_\infty}{(T_m - T_\infty)}, \quad Fr_s = \frac{v_s}{(g_m b_s)^{\frac{1}{2}}} \tag{41}$$

Where $b_s$, $v_s$ and $T_s$ are the width, velocity and temperature at the physical source note that we shall always define Froude numbers with respect to the constant buoyancy scale $g_m$ rather than the buoyancy of the plume, so that the Froude number is simply a dimensionless velocity. Given positive, finite values of $\varphi_s$ and $Fr_s$, we can find the corresponding co-ordinates in Q-M space as:

$$Q_s = \frac{1}{\phi_s} \tag{42}$$

And

$$M_s = (\frac{Fr_s^4 \lambda^2}{\phi_s^6})^{\frac{1}{5}} \tag{43}$$

Substituting (42) and (43) into equation (38) we have

$$M_0 = (\frac{5}{8\phi_s^3})^{\frac{2}{5}}[\frac{8\lambda Fr_s^2}{5} \pm (\phi_s - \phi_s^2)]^{\frac{2}{5}} \tag{44}$$

Where, any point $Q_s$, $M_s$ on a trajectory in Q-M space can be regarded as a possible physical source for a plume with

$$\phi_s = \frac{1}{Q_s}, \quad Fr_s = \frac{M_s^{\frac{5}{4}}}{\lambda^{\frac{1}{2}} Q_s^{\frac{3}{2}}} \tag{45}$$

## 3. Numerical Results

### 3.1 Descending Plumes

We have just carried out an investigation on some possible behaviours of downward plumes and results are shown in the various figures below. The results as shown in figure 2 are courses in the Q − M space for descending plumes with different values of $M_0$; were $M_0$ serves as the downward momentum flux. The plume's uniqueness as they are generated are in accordance with their respective $M_0$ values (i.e., according to whether $M_0 > or < (\frac{5}{2^5})^{\frac{2}{5}})$ with the source offering some sort of forcing that is depended on the downward momentum. For the sake of this study, the value $M_0 = (\frac{5}{2^5})^{\frac{2}{5}}$ is considered as the critical forcing term as we also have in the literature by Kay, 2007: and this leads to the fact that no rising plume can exist with $M_0 < -(\frac{5}{2^5})^{\frac{2}{5}}$. Similar to the nomenclature as used in the literature by Hunt & Kaye, 2005; any plume with $M_0$ value $> (\frac{5}{2^5})^{\frac{2}{5}}$ will be termed strongly forced plume. But then, for the interval $0 < M_0 < (\frac{5}{2^5})^{\frac{2}{5}}$, we are aimed at two different solutions: first is within the range $Q < \frac{1}{2}$ (such that φ > 2) this we call warm weakly forced plume: and when $Q > \frac{1}{2}$ (such that φ < 2) this we call cool weakly forced plume. Note that any plume that attains $\varphi = 1$ also correspond at the temperature of maximum density, and any part that corresponds to φ = 2 attains the same density as that of the ambient (due to (1)). Plumes such as pure and lazy may also exist in this part $Q > \frac{1}{2}$, being that warm plumes possess some sort of upward buoyancy in order for a downward movement. It is also necessary that there should be some sort of downward forcing from its emanating point. Results in figure 2 also showed that plumes in the cool part appears very similarly in the far field which corresponding to large Q.

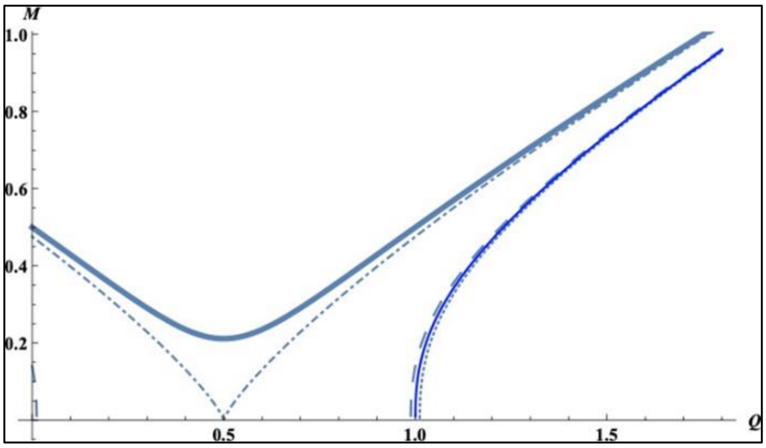


**Fig 2:** The courses pattern followed in Q-M space from solution (38) for downward motion, with M0 = −0.14 (dashed curve), M0 = 0 (thin solid curve), M0 = 0.14 (dotted curve), $M_0 = (\frac{5}{2^5})^{\frac{2}{5}}$ (dotted dashed curve) and M0 = 0.5 (bold solid curve). The (dotted dashed curve) for $M_0 = (\frac{5}{2^5})^{\frac{2}{5}}$ separates strongly forced from weakly forced plumes (i.e. at the centre $Q = \frac{1}{2}$ to the left hand side represents warm sector while, to the right hand side represents cool sector

The warm weakly forced plume possess an unlimited temperature at its virtual source and this supports the buoyancy force for an initial upward movement. This behaviour as just described appears different from previous results by Osaisai & George, 2024 [15] for an upward forced plum and it is reasonable. However, these results here on the other hand appears similar to those by Kay 2007 [12] in an earlier investigation which reported instead that "the behaviour near the source is similar to that for an upward forced plume". Ejection of the plume downwards provides an opposing buoyancy which in turn reduces the momentum flux as it continue to entrain the surrounding cold water. Here in this study, the term weakly forced implies: momentum flux at earlier time is not significant enough to stop the plume's movement before it cools down to the temperature of zero buoyancy (φ = 2). In a similar manner a strongly forced plume implies: the plume gets to the temperature of zero buoyancy but still having a positive downward momentum. Sometimes, one might also want to reason the possibility of a plume moving in a reversed direction which might be unphysical as also stated in the literature by Kay, 2007 [12] where the condition of infinite width will not only be met but that the plume will again follow the same space in the opposite direction. Though, a more realistic model to this behaviour have been reached by George & Kay, 2017 [9] for a fountain flow case, though, it will be inverted in the case of a descending weakly forced plume. It is also worth discussing the plume's motion in three distinct classes such as; strongly forced, warm weakly forced, and cool, using the critically forced plume as a limiting case of each class.

### 3.1.1. Strongly Forced Plumes

The results as shown in Figure 3 specifies the evolving process of a critically forced plume and that of the strongly forced plumes with mainly two values of M0. Result in panel (c) indicates the point of zero buoyancy and this is the point where momentum flux attains its minimum. Meanwhile, panel (a) indicates where the volume flux exceeds the value 21. Furthermore, the temperature seem to be fixed which implies a critically forced plume and as well the volume flux all being stable. And at this point the vertical velocity drops to zero enabling the plume to move a significant distance at a very reduced velocity with a very little buoyancy forces. The results in panel (d) for the strongly forced plumes showed that the vertical velocity possesses its minimum that goes beyond the point of zero buoyancy. Considering the far field behaviour where Q ≫ M0, M ~ Q from (38); and from equation (35) we have that both volume and momentum flux advances linearly with distance, such that the dimensionless velocity V approach unity as Y → ∞ in all the cases.

$$Y_n = \int_{Q_0}^{\frac{1}{2}} \frac{1}{2M^{\frac{1}{2}}} dQ \tag{46}$$

Figure 4 shows the result of the vertical distance $Y_n$ below the source where zero buoyancy occurs as expressed in equation

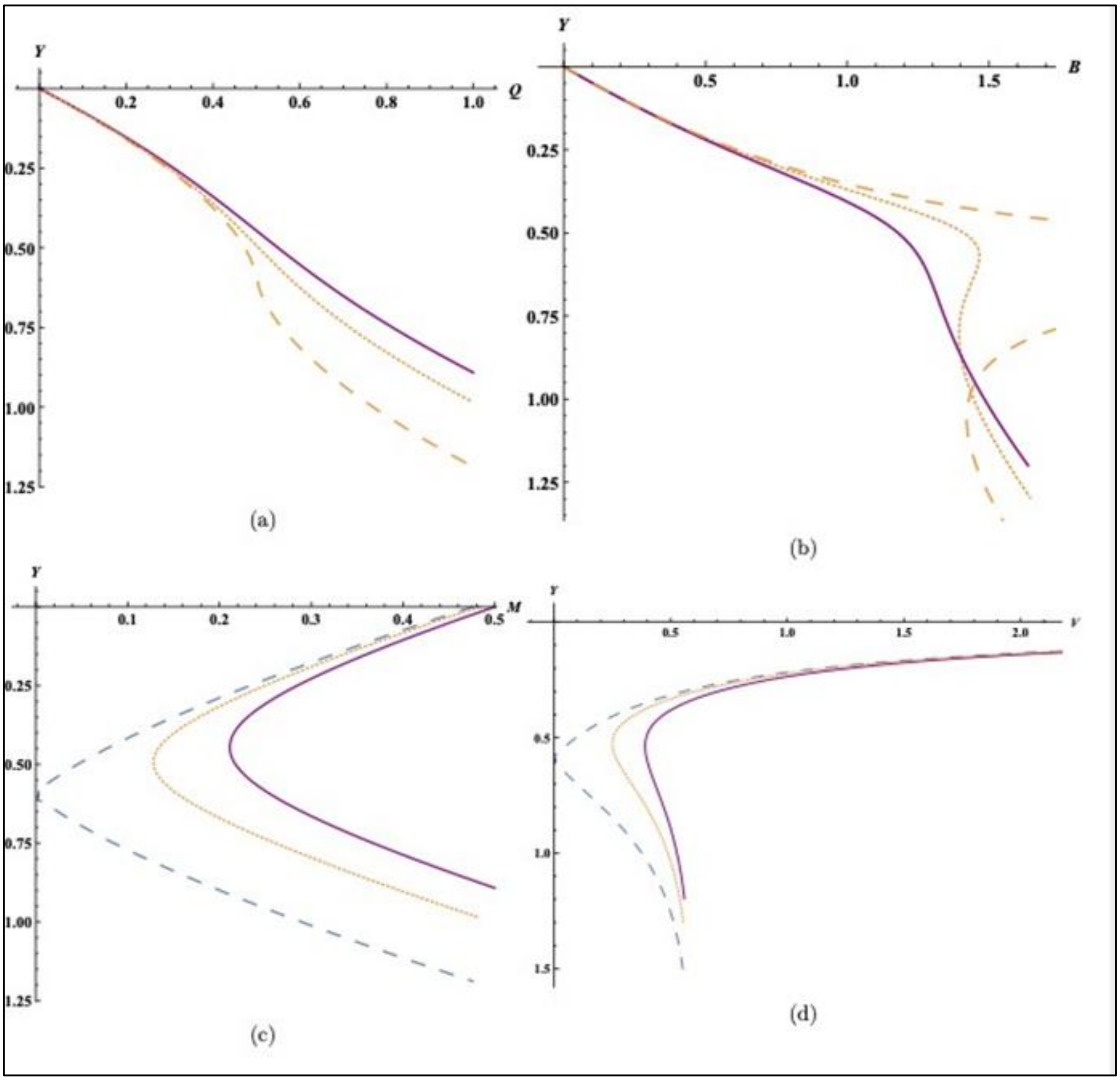


**Fig 3:** Dimensionless plume properties against vertical distance below an infinite-temperature virtual source for a critically forced plume with $M_0 = (\frac{5}{2^5})^{\frac{2}{5}} \approx 0.475913$, (dashed curves) and strongly forced plumes with $M_0 = 0.205$ (dotted curves) and $M_0 = 0.24$ (solid curves): (a) Volume flux, (b) Half-width, (c) Momentum flux, (d) Vertical velocity.

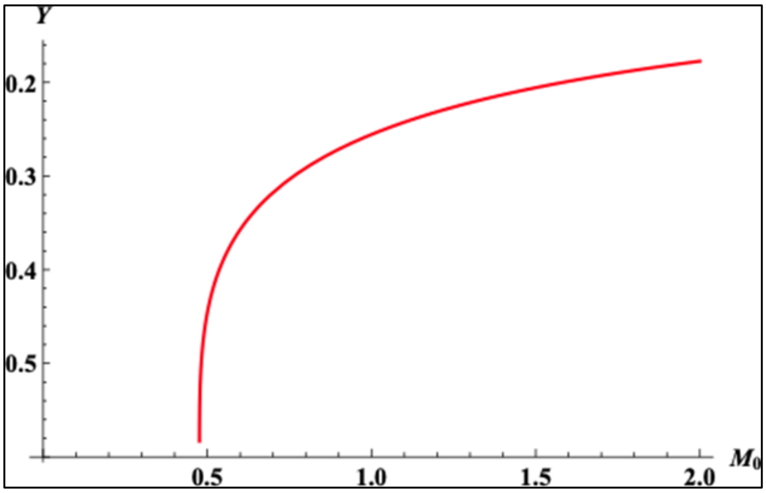


**Fig 4:** Distance downwards from source to point of zero buoyancy as a function of $M_0$ for strongly forced plumes.

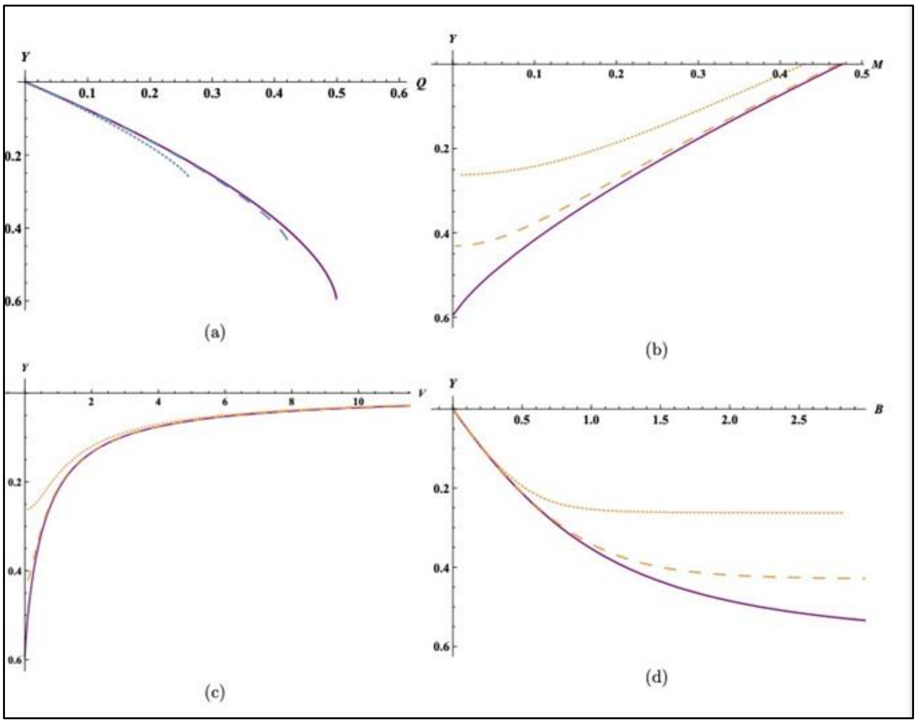


**Fig 5:** Dimensionless plume properties against vertical distance below an infinite-temperature virtual source for a critically forced plume (solid curves) and warm weakly forced plumes with $M_0 = 0.192$ (dashed curves) and $M_0 = 0.16$ (dotted curves): (a) Volume flux, (b) Momentum flux, (c) Vertical velocity, (d) Half-width

(46), where $Q_0 = 0$ and plotted against M0. It is also interesting to observe that $Y_n$ decreases rapidly even as M0 rises slightly above the critical value. Note, as recorded in the literature by Osaisai & George, 2024 [15] that greater velocity implies greater entrainment for rising plumes: this in turn causes a decrease in the distance travelled before zero-buoyancy is reached. Meanwhile, critically forced plume within the point of zero-buoyancy with some sort of reduced velocity and very little buoyancy forces shows different from a plume with greater velocity and buoyancy forces: and this is as a result of the distance travelled.

### 3.1.2. Warm weakly forced plumes

Results in Figure 5 shows the evolvement of a warm weakly forced plume for different values of $M_0$, and warm critically forced plume. The results showed that both warm weakly forced and critically forced plumes come to rest with final volume flux at $Q_f \leq \frac{1}{2}$ with an infinite width. However, a small departure from the critical forcing in our result does not really cause any major difference as compared to those by Kay, 2007; for which there was a big difference. The critically forced cases could move a lengthy distance with a reduced velocity, and this process enables low entrainment which in turn resulted to a moderately slow deceleration rate. But then, it is also worth noting that what we have just stated above can be systematically balanced; that is, a slight decrease in initial momentum flux from the critical value $(\frac{5}{2^5})^{\frac{2}{5}} \approx 0.475913$ may result to a huge reduction in total distance moved by the plume before coming to a halt. Figure 5 (b) also confirms this difference in the $Y_f$ as $M_0$ decreases from its critical value when compared with the Y – axis intercepts of the curves for $M_0 = (\frac{5}{2^5})^{\frac{2}{5}}$ and $M_0 = 0.192$. It is believed that when the momentum flux is not much at the beginning, then the warm descending plume will be brought to a halt very quickly through a strong opposing buoyancy force at high temperature so as for the plume to have entrained much cold water to reduce this force. With this, the plume may not move a far distance.

$$\phi_f = \frac{1}{Q_f} \tag{47}$$

It is also important to note that the temperature for which the warm weakly forced plume come to a halt is calculated using (47), with $Q_f$. The temperature here decreases monotonically with increasing initial momentum flux.

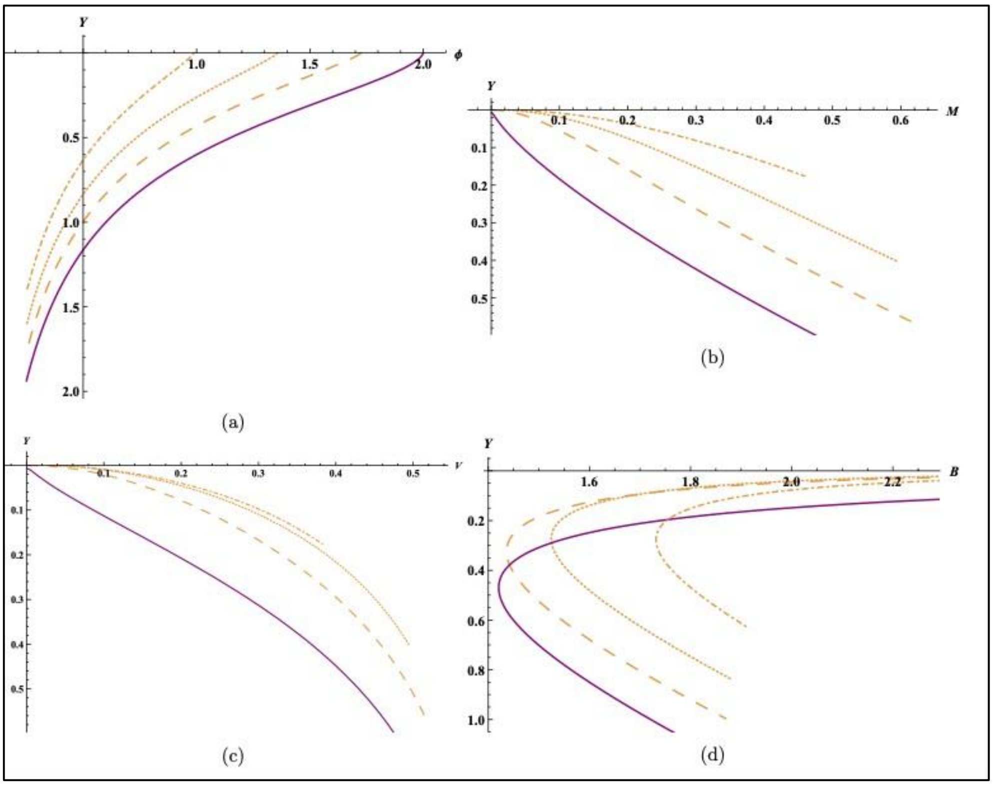


**Fig 6:** Dimensionless plume properties against vertical distance below an infinite-temperature virtual source for a cool critically forced plume (solid curves) and cool weakly forced plumes with $M_0$ = 0.192 (dashed curves) and $M_0$ = 0.16 (dotted curves) and lazy plume with $M_0$ = −0.14(dash-dotted curves): (a) Temperature, (b) Momentum flux, (c) Vertical velocity, (d) Half-width.

### 3.2. Cool Plumes

The results as shown in this section arises from a virtual source with finite temperature, volume flux $Q_0 \geq \frac{1}{2}$ and zero momentum flux. The M0 values as considered for these plume are also in line with equation (38) with Q = $Q_0$ and M = 0. However, it will not be proper to consider the same as initial momentum flux because it would need an un- physical continuation from a theoretical point of view. If we consider the source of a critically forced plume as the point where the plume is at rest with $Q = \frac{1}{2}$. Then, the results as presented in figure 6 explains the evolution of a critically forced plume where two of the plumes are weakly forced with smaller values of $M_0$ as those in Fig. 5: meanwhile, one of the plumes is a lazy plume with $M_0$ value −0.14. But then, despite the difference in their M0 values, the behavior in both the weakly forced and lazy plumes appears very similar, but slightly different from that of the critically forced plume. At some point away from the source, it was observed that gain in momentum flux of the critically forced plume, acceleration from rest and reduction in temperature are all delayed which is as a function of the distance from source (see Fig. 6). Descending cool plumes also possess some sort of infinite width at their source: but then, becomes wider at some distance away from the source. Thus, the plume's width will attain its minimum value at some point. Plume's temperature at some distance away from the source is very close to that of the ambient temperature than the $T_m$, such that the nonlinearity of equation (1) is a little correction if the interaction of the plume with the ambient is of interest.

The result as shown in figure 2 also suggest that all plumes in the far field seem to have rising from a finite-temperature virtual source: where this contradict the linear dependent cases because, locating a specific position at the source for all plumes is not possible.

All the results case as presented here are also possible behaviours of downward plumes in a fluid with the quadratic dependent relation assumption. Though, the plumes are assumed to have come from a virtual source condition where giving an interpretation as though they were from physical sources may be difficult. In terms of the study of plumes from physical sources, there are two types of physical source which might be of practical importance. One of such sources is a power station warm discharge at a lake floor which will result to having upward buoyancy and upward vertical velocity: this we consider to be beyond the scope of this present study. Next is a plume that descends from a surface gravity current, the fluid in this region is water that have mixed up to $T_m$, or to a temperature close to it: or the point of zero buoyancy with dimensionless temperature $\varphi_s$ below 2, volume flux $Q_s$ above $\frac{1}{2}$, momentum flux Ms and Froude number.

$F_{rs}$ which is small but not negative. This is a cool descending plume as stated in section 3 with detailed description. Though, those results in section 3.2 requires that there is the necessity to prevent singularity at the virtual source (see Fig. 6) by observing that the initial conditions suggest a physical source position at some point below the virtual source. Cool plumes are known to be the only descending plume that does not need to eject a fluid downwards against the buoyancy force from its source. Equation (33) may be considered appropriate for the power station cooling waste water discharge problem; where temperatures of both the ambient and the discharge water, volume flux of the warm water discharged are fixed according to the environmental conditions and the power station standards so that both scales of $_{qT}$ and $_{gm}$ are also fixed. On the other hand, it might also be true that these parameters are in a way unclear for practical purposes. Especially, with a physical source of half-width $b_s$ it would be a good idea to define dimensionless heights with respect to this parameter. This we also believe is beyond the scope of this investigation.

## 4. Discussion/Conclusion

Downward axisymmetric plumes with their possible behaviours in a quiescent ambient have just been studied numerically, with the assumption that density is a quadratic function of temperature. Results here are for different values of $M_0$, where the value $M_0 = (\frac{5}{2^5})^{\frac{2}{5}}$ is termed critical forcing. Plumes with $M_0$ value > $(\frac{5}{2^5})^{\frac{2}{5}}$ were termed strongly forced plume. Within the interval 0 < $M_0 < (\frac{5}{2^5})^{\frac{2}{5}}$, two solutions are aimed: a warm weakly forced and cool weakly forced plumes for Q < $\frac{1}{2}$ and Q > $\frac{1}{2}$ respectively. Plume's uniqueness are in accordance with their respective M0 values with the source offering some sort of forcing that is depended on the downward momentum. The warm weakly forced plume possess an unlimited temperature at its virtual source, which supports the buoyancy force for an initial upward movement. A similar behaviour was also recorded by Kay, 2007: but then, contrary from previous results by Osaisai & George, 2024 [15] for rising plumes. The evolving process and point of zero buoyancy are also shown for the critically forced and strongly forced plumes in Figure 3, where both temperature and volume flux are stable at this point; but then, the vertical velocity drops to zero enabling the plume to move a distance with very little buoyancy forces. At far field where $Q \gg M_0$, $M \sim Q$ in (38) and from equation (35), both volume and momentum flux advances linearly with distance, such that the dimensionless velocity V approach unity as $Y \rightarrow \infty$. As recorded in the literature by Osaisai & George, 2024 [15]; greater velocity implies greater entrainment for rising plumes, resulting to a decrease in the distance travelled before attaining zero-buoyancy. Meanwhile, critically forced plume at the point of zero-buoyancy with small velocity and very little buoyancy forces shows different from a plume with greater velocity and buoyancy forces and this is evident in the distance travelled. Our results also showed that both warm weakly forced and critically forced plumes come to rest with final volume flux at $Q_f \leq \frac{1}{2}$ with an infinite width. A small departure from the critical forcing in our result does not really cause any major difference as compared to those by Kay, 2007. A confirmation to this difference in the $Y_f$, as $M_0$ decreases from its critical value when compared with the Y − axis intercepts of the curves for $M_0 = (\frac{5}{2^5})^{\frac{2}{5}}$ and $M_0 = 0.192$. The temperature here decreases monotonically with increasing initial momentum flux. Our plumes are assumed to have come from a virtual source condition where giving interpretation as though they were from physical sources may be difficult. In terms of the study of plumes from physical sources, two types of such might be of practical importance: one of which is the power station warm discharge at a lake floor and a plume that descends from a surface gravity current. Fluid that can be found in the later is water that have mixed up to Tm, volume flux Qs above 21, momentum flux Ms and Froude number $F_{rs}$ which is small but not negative. Results in section 3.2 requires that there is the necessity to prevent singularity at the virtual source (see Fig. 6) by ensuring that the initial conditions suggest a physical source position at some point below the virtual source and this we take as a limitation. As also stated in the literature by Osaisai & George, 2024 [15] that a more realistic model had been reached by George & Kay, 2017 for a fountain flow case, but need to be inverted in the case of a descending weakly forced plume. Lastly, with the entrainment model, it is required that the plume will be fully turbulent, a condition usually obtained with high Reynolds number (Fischer *et al*. 1979) [3].
Whereas, our case here can be likened to the power station warm discharge which will definitely be fully turbulent with its large volume flux. These are basic conditions on the validity of these assumptions. In overall, the numerical results here are very good as they present us with insight into downward plumes with the quadratic dependence relation assumption.

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