On the Robustness to Unmatched Disturbances of Finite-Time Dynamic Surface Control (FT-DSC): A Comparison with Second-Order TSM in a Third-Order System

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Abstract
Controlling high-order nonlinear systems in the presence of disturbances—especially unmatched disturbances (those acting at channels different from the control input)—is a challenging problem. For third-order systems, the system structure itself creates additional difficulties in designing controllers that guarantee both performance and stability. Dynamic Surface Control (DSC) is a recursive backstepping-based technique that circumvents the derivative-explosion problem of traditional backstepping by introducing first-order filters. When nonlinear (exponential) terms are used inside these filters, they form finite-time systems without incurring the computational burden of high-order derivatives. Such finite-time DSC (FT-DSC) not only preserves the advantages of standard DSC but also enhances its rejection of mismatched disturbances compared with conventional finite-time sliding controllers.

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Introduction

Dynamic Surface Control (DSC), first introduced by Wang & Sun (1999), addresses the explosion of complexity inherent in backstepping by employing low-order filters rather than repeated differentiation [1]. This idea has become the cornerstone of many subsequent extensions, particularly for systems with strong nonlinearities or large disturbances.

Recent studies have applied DSC to relatively simple actuators such as DC motors—ubiquitous in experimental practice. Shiledar & Malwatkar (2025) combined DSC with a discrete sliding reach law to control DC-motor speed in a disturbed environment and demonstrated superior stability and reduced oscillation over classical SMC [2]. Aydın & Yakut (2024) (though not using DSC directly) modeled and controlled a rotary inverted pendulum driven by a DC motor, providing a solid baseline for integrating DSC [3]. Kurczak *et al*. (2024) presented a laboratory hardware platform based on a DC motor that is ideal for testing DSC algorithms [4].

Beyond these fundamentals, many DSC variants have been proposed, including adaptive and finite-time versions. Li *et al*. (2025) integrated DSC with adaptive neural networks for stochastic pure-feedback systems with incomplete measurements [5]. Han & Feng (2025) incorporated Barrier Lyapunov Functions to handle state constraints [6]. Yu *et al*. (2025) [7] and Shi *et al*. (2025) [8] promoted predefined-time / prescribed-performance DSC, eliminating dependency on initial conditions. Other work—for example Agrawal & Misra (2025) [9] and Kang *et al*. (2025) [10]—extended DSC to systems under faults or cyber-attacks (FDI), underscoring its flexibility under complex uncertainties.

However, the inherent disturbance-rejection capability of DSC—especially finite-time DSC employing exponential terms—against mismatched disturbances has not been systematically evaluated. This paper therefore investigates and directly compares the robustness of finite-time DSC (FT-DSC) with that of second-order Terminal Sliding Mode (TSM), proposed by Huspeka (2009) [11], on a third-order system subjected to both matched and mismatched disturbances.

Methodology

A. Conventional Second-Order TSM for a DC Motor

1. **DC- Motor model**

The electrical and mechanical dynamics are described by equation (1).

$$\dot{\boldsymbol{x}} = \begin{bmatrix} \dot{\delta}_i \\ \dot{\omega}_i \\ i_i \end{bmatrix} = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{B}u + \boldsymbol{D} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\frac{b}{J} & \frac{k_m}{J} \\ 0 & -\frac{k_e}{L} & -\frac{R}{L} \end{bmatrix} \begin{bmatrix} \delta_i \\ \omega_i \\ i_i \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{L} \end{bmatrix} u + \begin{bmatrix} 0 \\ d_2 \\ d_3 \end{bmatrix} \quad (1)$$

Where δ_i is the rotor angle, ω_i is the angular velocity, i_i is the armature current, and the input u is the armature voltage. Parameters R, L are the armature resistance and inductance, k_e is the back-EMF constant, b the viscous friction coefficient, J the rotor inertia, and k_m the torque constant.

- d_2: represents unmatched disturbance
- d_3: is matched disturbance.

2. TSM bậc 2

The coordinate transformation, error definitions, sliding variables, and control law formulation follow Huspeka [11]. Then, equation (1) is needed to be transformed by matrix T:

$$\boldsymbol{x_f} = \boldsymbol{T}\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{b}{J} & \frac{k_m}{J} \end{bmatrix} \boldsymbol{x} \quad (2)$$

Then (1) becomes:

$$\dot{\boldsymbol{x}}_{\boldsymbol{f}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\frac{k_m k_e + bR}{JL} & -\frac{JR + bL}{JL} \end{bmatrix} \begin{bmatrix} x_{1,f} \\ x_{2,f} \\ x_{3,f} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{k_m}{JL} \end{bmatrix} u + \begin{bmatrix} 0 \\ d_2 \\ d_3 \end{bmatrix} \quad (3)$$

System (3) is in nominal form. This is the dynamic model for DC Motor. Set:

$$\begin{cases} a_2 = \frac{k_m k_e + bR}{JL} \\ a_3 = \frac{JR + bL}{JL} \\ b_3 = \frac{k_m}{JL} \end{cases} \quad (4)$$

When the DC Motor follows a reference value, system (3) is needed to transform to error form.

Set: $e_1 = x_{1,f} - x_{1d}$ (5)

Then by (3), we have:

$$\begin{cases} e_2 = x_{2,f} - x_{2d} = \dot{x}_{1,f} - \dot{x}_{1d} = \dot{e}_1 \\ e_3 = x_{3,f} - x_{3d} = \dot{x}_{2,f} - \dot{x}_{2d} + \frac{T_L}{J} = \dot{e}_2 + \frac{T_L}{J} \\ \dot{e}_3 = -a_2 e_2 - a_3 e_3 + b_3 u + a_2 x_{2d} + a_3 x_{3d} - \dot{x}_{3d} \end{cases} \quad (6)$$

Set: $D_M = a_2 x_{2d} + a_3 x_{3d} - \dot{x}_{3d}$ (7)

Then the error form of dynamic equation is:

$$\dot{\boldsymbol{e}} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -a_2 & -a_3 \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ b_3 \end{bmatrix} u + \begin{bmatrix} 0 \\ d_2 \\ D_M + d_3 \end{bmatrix} \quad (8)$$

To apply Cascaded TSM for (8), a secondary sliding

$$\sigma = c_1 e_1 + c_2 e_2 \quad (9)$$

Where c_1, c_2 >0 are predefined parameters.

Then its derivatives

$$\dot{\sigma} = c_1\dot{e}_1 + c_2\dot{e}_2 = c_1e_2 + c_2e_3 - c_2d_2 \quad (10)$$

$$\ddot{\sigma} = c_1\dot{e}_2 + c_2\dot{e}_3 = c_1e_3 - c_1d_2 + c_2(-a_2e_2 - a_3e_3 + b_3u + D_M + d_3) \quad (11)$$

Then the primary sliding variable is

$$S = c\sigma^{\frac{p}{q}} + \dot{\sigma}\ , c>0 \quad (12)$$

Where p, q are odd numbers, such that $\frac{p}{q}$ is close to 0.5 (the chosen values are p =1001 and q = 2001).
Then S's derivative:

$$\dot{S} = c\frac{p}{q}\sigma^{\frac{p-q}{q}}\dot{\sigma} + \ddot{\sigma} \quad (13)$$

Then the conventional TSM sliding controller has two components: a discrete component that pulls the sliding variable to the sliding surface and a continuous component that keeps the sliding variable on the sliding surface:

$$u = u_{eq} + u_d \quad (14)$$

Then with the control input value u_{eq} applied to (11) and (13) we will have:

$$\dot{S} = 0 \quad (15)$$

From here we can find u_{eq}

$$u_{eq} = \frac{-1}{c_2b_3}(c\frac{p}{q}\sigma^{\frac{p-q}{q}}\dot{\sigma} + [c_1e_3 - c_2(a_2e_2 + a_3e_3 - D_M - d_3)] - c_1d_2) \quad (16)$$

Where, the values d_2, d_3 may not be determinable and are replaced by 0.
When the control input is u for equation (14), the continuous control component u_{eq} will neutralize most of the components in the expression of $\dot{S}$:

$$\dot{S} = c_2b_3u_d \quad (17)$$

Then with the chosen Lyapunov candidate function:

$$V = \frac{1}{2}S^2 \quad (18)$$

Then the derivative of V

$$\dot{V} = S\dot{S} = c_2b_3u_dS \quad (19)$$

Since we always have $cc_2b_3 > 0$, u_d can be chosen so that $\dot{V} \leq 0$

$$u_d = -K\text{sign}(S) \quad (20)$$

Then with the control input chosen according to (14), (16), and (20), the system is Lyapunov stable.

B. Constructing the Finite-Time Dynamic Sliding Surface Controller (FTDSC):

1. **FTDSC Control**: The DSC controller is designed to avoid the "explosion of complexity" problem of traditional backstepping by using finite-time filters. The surface errors are defined as follows:

$$\begin{cases} s_1 = x_1 - y_d \\ s_2 = x_2 - z_1 \\ s_3 = x_3 - z_2 \end{cases} \quad (21)$$

Where, z_1, z_2 are the states of the filter, approximating the virtual control functions α_1 and α_2.
Virtual control function α_1

$$\alpha_1 = -k_1 s_1^{\frac{p}{q}} - k_2 s_1 + \dot{y}_d \quad (22)$$

Finite-time filter for z_1

$$\dot{z}_1 = -\frac{1}{\tau_1}(z_1 - \alpha_1) - k_f (z_1 - \alpha_1)^{\frac{p}{q}} \quad (23)$$

Virtual control function α_2

$$\alpha_2 = \frac{J}{k_m}\left(-k_3 s_2^{\frac{p}{q}} - k_4 s_2 + \frac{b}{J} x_2 + \dot{z}_1\right) \quad (24)$$

Finite-time filter for z_2

$$\dot{z}_2 = -\frac{1}{\tau_2}(z_2 - \alpha_2) - k_f (z_2 - \alpha_2)^{\frac{p}{q}} \quad (25)$$

Control law u

$$u = L\left(-k_5 s_3^{\frac{p}{q}} - k_6 s_3 + \frac{R}{L} x_3 + \frac{k_e}{L} x_2 + \dot{z}_2\right) \quad (26)$$

2. Stability and Finite-Time Convergence Analysis:

Finite-time means that the tracking error $s_1 = x_1 - y_d$ converges to 0 or a small neighborhood within a finite time, despite disturbances d_2, d_3. We will analyze each surface error s_1, s_2, s_3 and the filters z_1, z_2 sing Lyapunov functions.

Error s_1

Lyapunov function with s_1

$$V_1 = \frac{1}{2} s_1^2 \quad (27)$$

Derivative: $\dot{V}_1 = s_1 \dot{s}_1 = s_1(\dot{x}_1 - \dot{y}_d) = s_1(x_2 - \dot{y}_d)$ (28)

Since $x_2 = s_2 + z_1$ then:

$$\dot{V}_1 = s_1(s_2 + z_1 - \dot{y}_d) \quad (29)$$

With the filter z_1 defined by (23), we can define the filter error:

$$e_{z1} = z_1 - \alpha_1, e_{z2} = z_2 - \alpha_2 \quad (30)$$

Then combining (29) with (30) and (22) we have:

$$\dot{V}_1 = s_1\left(s_2 + e_{z1} - k_1 s_1^{\frac{p}{q}} - k_2 s_1\right) = s_1 s_2 + s_1 e_{z1} - k_1 s_1^{\frac{p}{q}+1} - k_2 s_1^2$$

Lyapunov function for e_{z1}

$$V_{z1} = \frac{1}{2} e_{z1}^2 \quad (31)$$

Derivative:

$$\dot{V}_{z1} = e_{z1} \dot{e}_{z1} = e_{z1}(\dot{z}_1 - \dot{\alpha}_1) \quad (32)$$

Combining with (23), we have:

$$\dot{V}_{z1} = e_{z1}\left(-\frac{1}{\tau_1} e_{z1} - k_f e_{z1}^{\frac{p}{q}} - \dot{\alpha}_1\right) = -\frac{1}{\tau_1} e_{z1}^2 - k_f e_{z1}^{\frac{p}{q}+1} - e_{z1}\dot{\alpha}_1 \quad (33)$$

Error s_2

Lyapunov function with s_2

$$V_2 = \frac{1}{2}s_2^2 \tag{34}$$

$$\text{Derivative: } \dot{V}_2 = s_2\dot{s}_2 = s_2(\dot{x}_2 - \dot{z}_1) \tag{35}$$

Combining with equation (1), we have

$$\dot{V}_2 = s_2\left(-\frac{b}{J}x_2 + \frac{k_m}{J}x_3 + d_2 - \dot{z}_1\right) \tag{36}$$

Combining with equations (21), (24), and (30), and assuming $z_2 \to \alpha_2$, we have:

$$\frac{k_m}{J}x_3 = \frac{k_m}{J}(s_3 + z_2) \approx \frac{k_m}{J}s_3 - k_3 s_2^{\frac{p}{q}} - k_4 s_2 + \frac{b}{J}x_2 + \dot{z}_1 \tag{37}$$

Then (36) becomes

$$\dot{V}_2 = \frac{k_m}{J}s_2 s_3 - k_3 s_2^{\frac{p}{q}+1} - k_4 s_2^2 + s_2 d_2 \tag{38}$$

Similarly if

$$V_3 = \frac{1}{2}s_3^2 \tag{39}$$

Then its derivative is calculated by

$$\dot{V}_3 = -k_5 s_3^{\frac{p}{q}+1} - k_6 s_3^2 + s_3 d_3 \tag{40}$$

Lyapunov function for e_{z2}

$$V_{z2} = \frac{1}{2}e_{z2}^2 \tag{41}$$

Derivative

$$\dot{V}_{z2} = e_{z2}\dot{e}_{z2} = -\frac{1}{\tau_2}e_{z2}^2 - k_f e_{z2}^{\frac{p}{q}+1} - e_{z2}\dot{\alpha}_2 \tag{42}$$

Sum of their derivatives

$$\dot{V} = s_1 s_2 + s_1 e_{z1} - k_1 s_1^{\frac{p}{q}+1} - k_2 s_1^2 + \frac{k_m}{J}s_2 s_3 - k_3 s_2^{\frac{p}{q}+1} - k_4 s_2^2 + s_2 d_2 - k_5 s_3^{\frac{p}{q}+1} - k_6 s_3^2 + s_3 d_3 - \frac{1}{\tau_1}e_{z1}^2 - k_f e_{z1}^{\frac{p}{q}+1} - e_{z1}\dot{\alpha}_1 - \frac{1}{\tau_2}e_{z2}^2 - k_f e_{z2}^{\frac{p}{q}+1} - e_{z2}\dot{\alpha}_2 \tag{43}$$

Applying Cauchy-Schwarz inequality for cross terms

$$s_1 s_2 \le \frac{1}{2}s_1^2 + \frac{1}{2}s_2^2 \tag{44}$$

$$s_1 e_1 \le \frac{1}{2}s_1^2 + \frac{1}{2}e_1^2 \tag{45}$$

$$\frac{k_m}{J}s_2 s_3 \le \frac{1}{2}\left(\frac{k_m}{J}\right)^2 s_2^2 + \frac{1}{2}s_3^2 \tag{46}$$

Bounded disturbance terms

$$|s_2 d_2| \le |s_2||d_2| \le |s_2||D_2| \tag{47}$$

$$|s_3 d_3| \le |s_3||d_3| \le |s_3||D_3| \tag{48}$$

Usually $\dot{\alpha}_1, \dot{\alpha}_2$ are also bounded because the required values and states are usually bounded, assume $|\dot{\alpha}_1| \le M_1$ và $|\dot{\alpha}_2| \le M_2$, then:

$$|e_{z1}\dot{\alpha}_1| \le |e_{z1}|M_1 \tag{49}$$

$$|e_{z2}\dot{\alpha}_2| \le |e_{z2}|M_2 \tag{50}$$

Then if $|s_2| > \left(\frac{D_2}{k_3}\right)^{\frac{q}{q+p}}$ we will have

$$-k_3 s_2^{\frac{p}{q}+1} + |s_2|D_2 < 0 \tag{51}$$

Similarly with other component expressions we will have

$$\dot{V} < 0 \tag{52}$$

Thus, by choosing appropriate coefficients, it is always possible to bring the deviations s_1, s_2, s_3 close to the neighborhood of 0 or the system is stable in a neighborhood around the origin with a size proportional to the amplitudes D_2, D_3, M_1, M_2.

C. Simulation

The simulation process was performed in the Matlab Simulink environment with the parameters given in (53):

$$\begin{cases} J = 0.01 \\ b = 0.1 \\ k_m = 0.01 \\ k_e = 0.01 \\ R = 1 \\ L = 0.5 \end{cases} \tag{53}$$

The desired value is considered a sinusoidal signal given by the following equations:

$$\begin{cases} x_d = \sin(t) \\ \dot{x}_d = \cos(t) \\ \ddot{x}_d = -\sin(t) \end{cases} \tag{54}$$

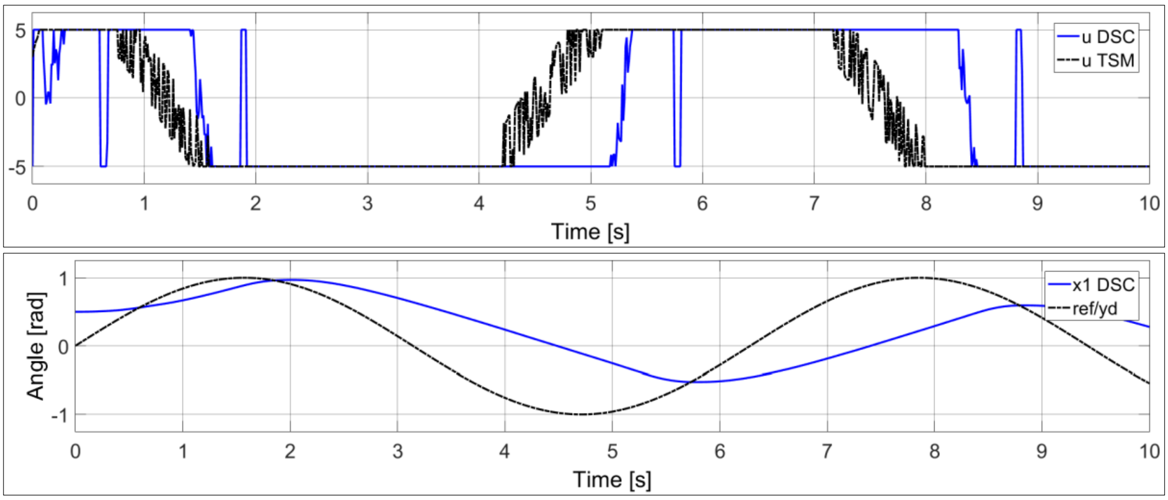
The disturbance acting on the system is considered bounded with amplitudes D_2, D_3:

$$\begin{cases} D_2 = 0.5 \\ D_3 = 0.75 \end{cases} \tag{55}$$

To compare the two control methods DSC and 2nd order TSM, choose the control input constraint according to different values: 5, 10, 15, 30.

KẾT QUẢ VÀ THẢO LUẬN

The simulation results are shown in Figure 1 for the case where the control input is limited by 5.



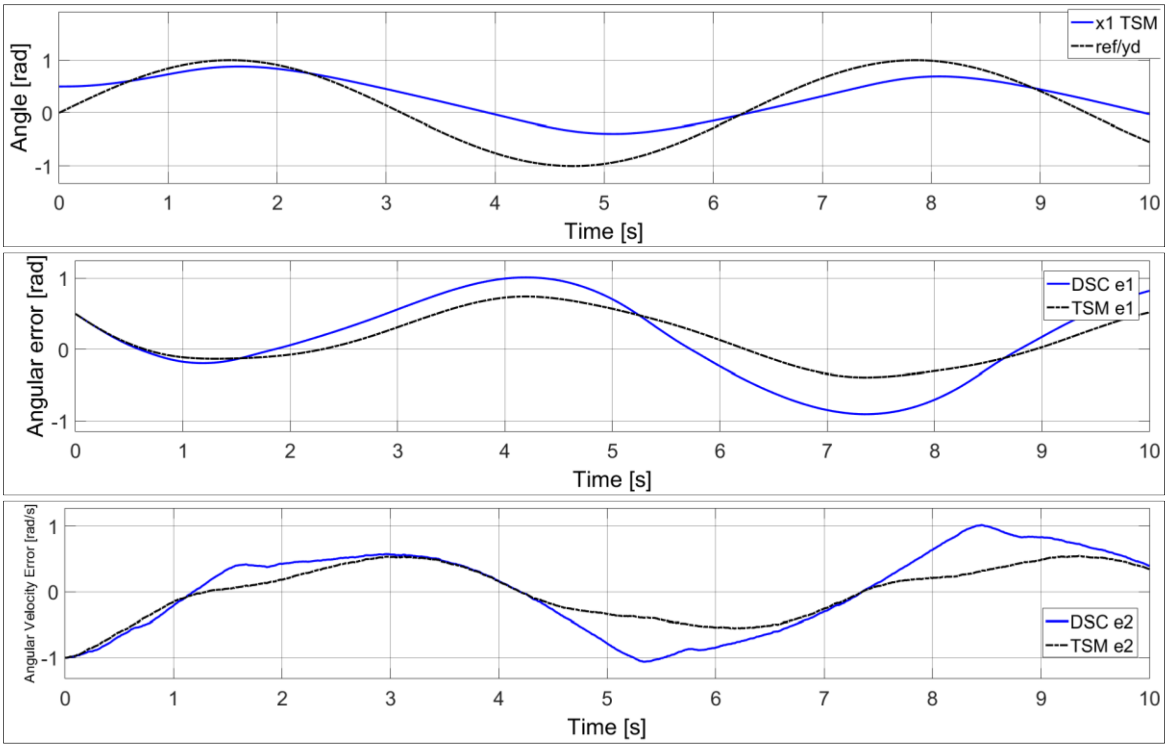
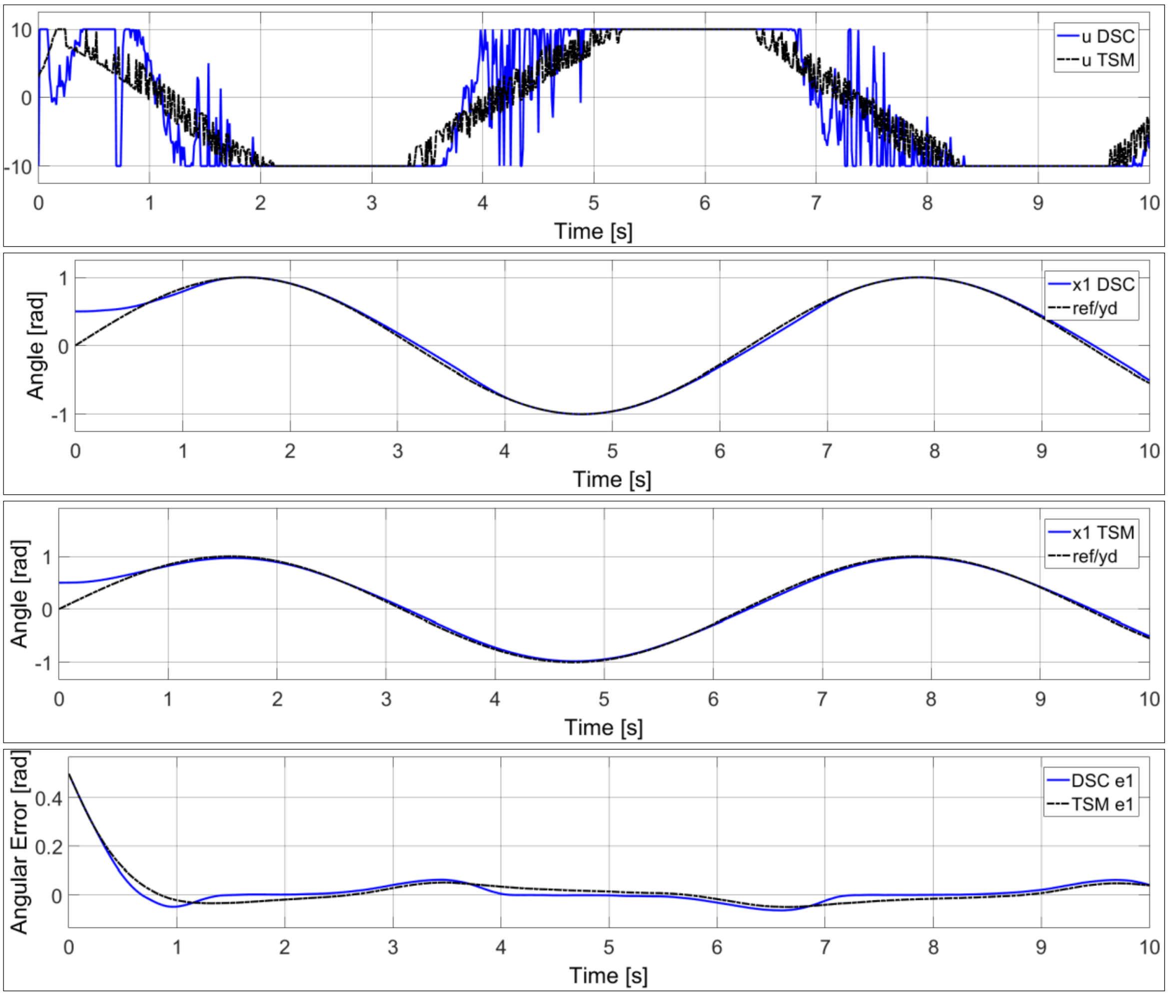


Fig 1: Simulation results when the control input is limited by 5.

The simulation is shown in Figure 2 for the case where the control input is limited by 10.



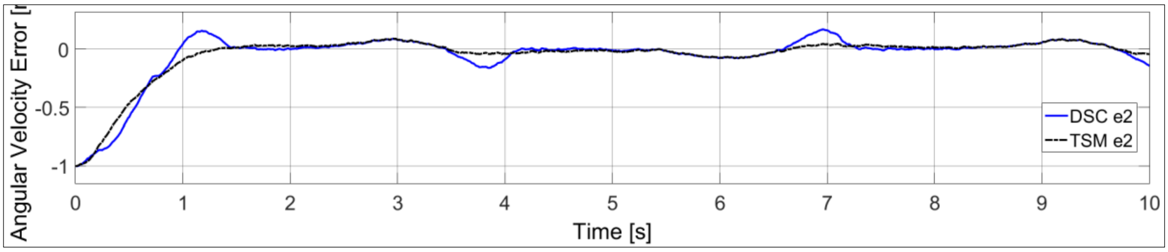


Fig 2: Simulation results when the control input is limited by 10.

The simulation is shown in Figure 3 for the case where the control input is limited by 15.

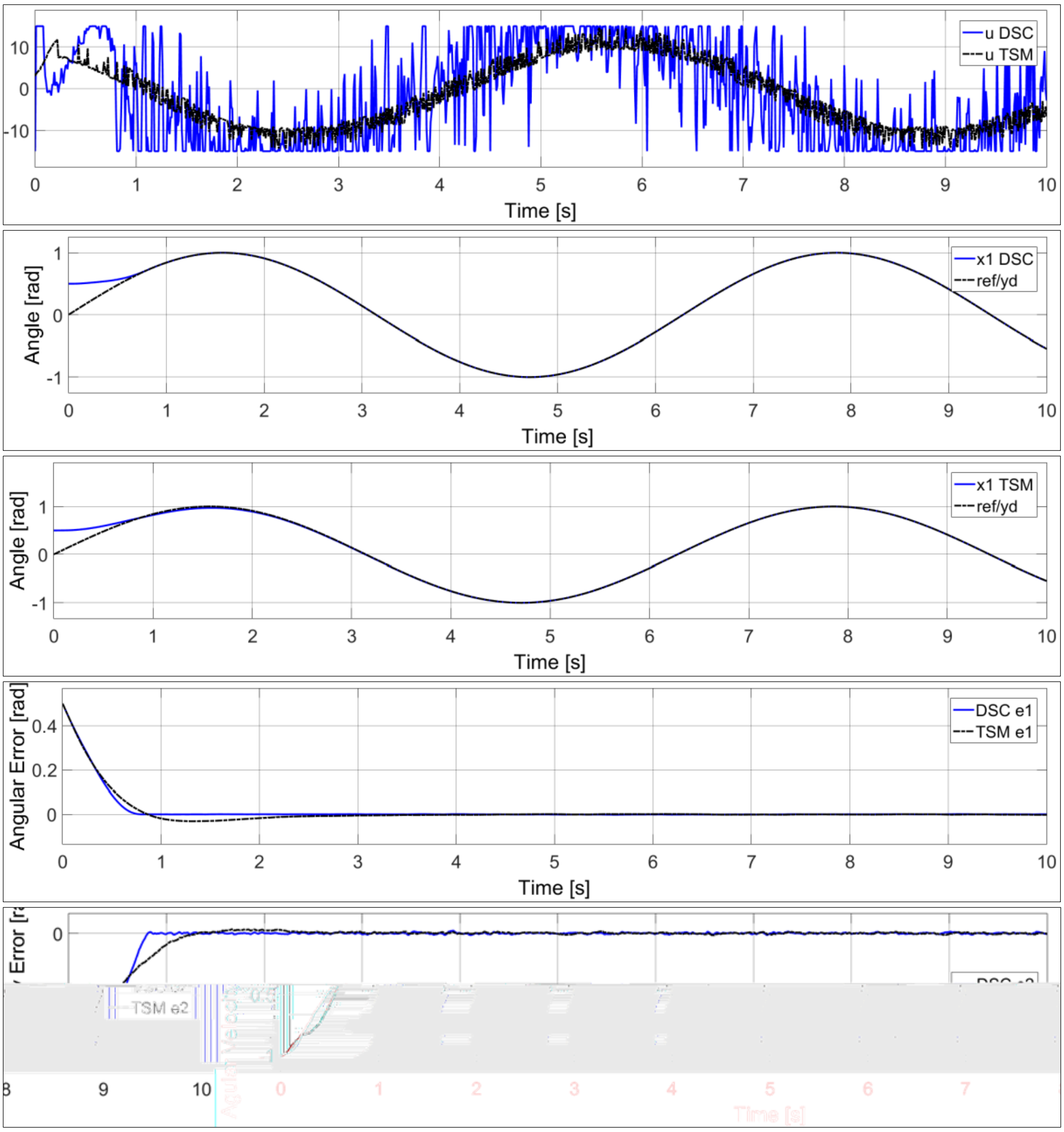


Fig 3: Simulation results when the control input is limited by 15.

The simulation is shown in Figure 4 for the case where the control input is limited by 30.

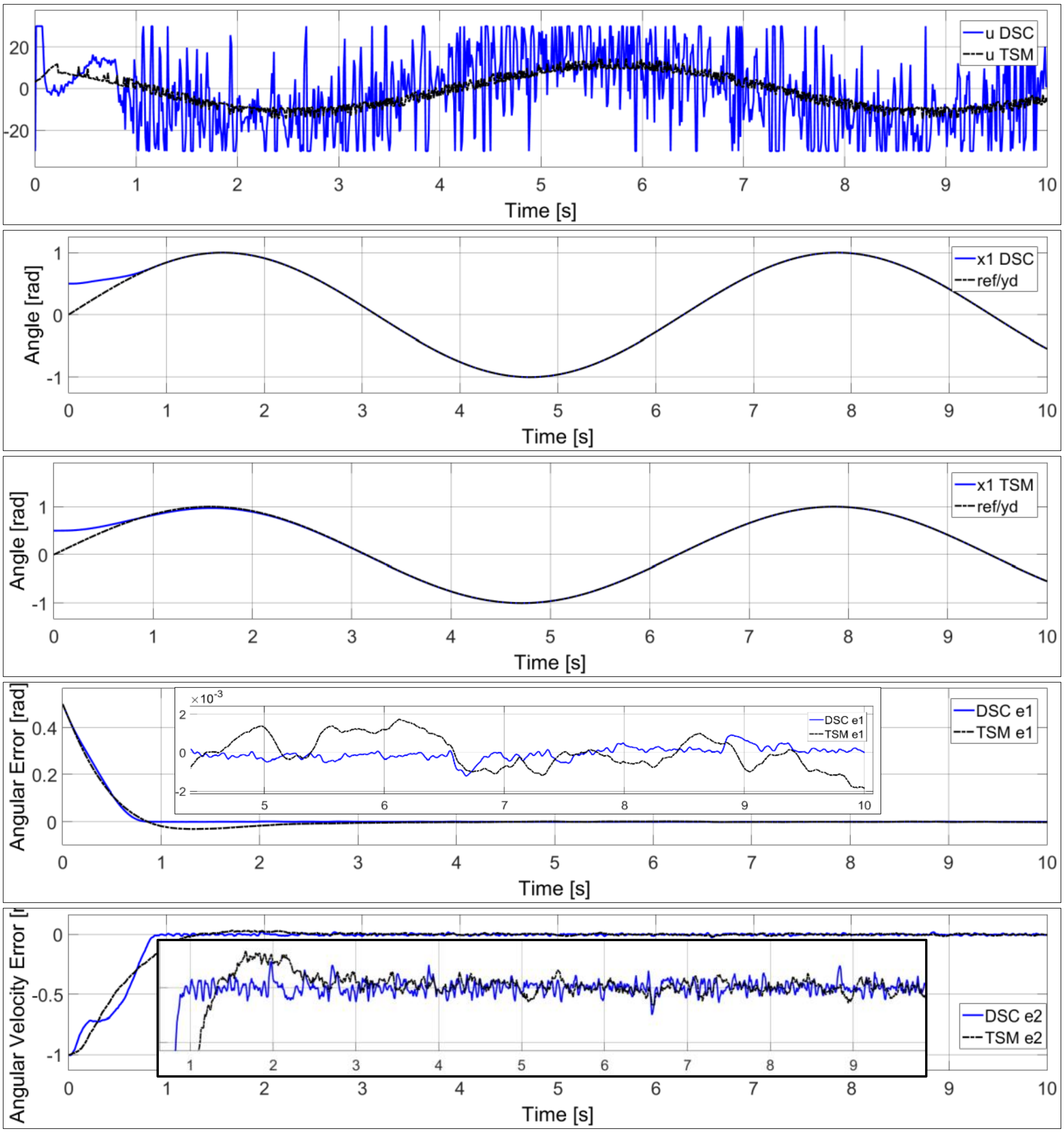


Fig 4: Simulation results when the control input is limited by 30.

The results in Figure 1 and Figure 2 show cases where the control input is insufficient for the energy demand of DSC. In that case, the ability of DSC to track the desired value is worse than that of 2nd order TSM, especially the angular velocity deviation. When the cases in Figure 3 and Figure 4, the control input is increased, although not enough to ensure according to the requirements of the control command, the trajectory tracking deviation of DSC has increased significantly, in which the fast convergence time of the system compared to 2nd order TSM is prominent, especially in Figure 4 when the convergence time of the system using DSC is less than 1s while with 2nd order TSM it is 3s. In that case, the quality of the system in those cases is also different. In Figure 3, the disturbance rejection capability of the two methods is almost equivalent, while with the ability to resist both matched and mismatched disturbances, we see in Figure 4, the disturbance in the DSC system has been significantly reduced compared to 2nd order TSM.

However, the effect of DSC in disturbance rejection for the velocity channel is not superior to 2nd order TSM when in all 4 cases the velocity tracking deviation of DSC is not better than 2nd order TSM.

Conclusion

This article has compared the DSC method with 2nd order TSM. The simulation, comparison, and evaluation results have pointed out the advantages and disadvantages of the two methods, which are:

- The DSC method has the advantage of minimizing the impact of disturbances on the system compared to 2nd order TSM.
- The DSC method requires a larger control input than the 2nd order TSM method. However, even when the control input is not sufficient to best meet the DSC system, it still converges faster and its disturbance rejection capability is equivalent to 2nd order TSM.
- When increasing the control input to a certain limit close enough to meet the demand of DSC, the system quality improves significantly compared to 2nd order TSM. The superior disturbance rejection capability is accompanied by a significantly

shorter convergence time. Therefore, considered comprehensively, the DSC controller has diverse application potential and better tracking quality of the desired value in conditions requiring fast system response time as well as ensuring the ability to resist both matched and mismatched disturbances.

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