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Common Fixed Point Theorems and Non-Expansive Mapping In Banach Space

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Abstract
In this paper, we describe non-expansive mapping in Banach space and several popular fixed point theorems. Our goal is to apply the non-expansive mapping and theorems to Banach space.

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1. Introduction
In this chapter, we describe non-expansive mapping in Banach space and several popular fixed point theorems. Our goal is to apply the non-expansive mapping and theorems to Banach space. We generalize a well-known Gregus (1980) result by establishing a common fixed point theorem for self-mappings that are not always commuting of a closed and convex subset of a Banach space. For every α, β in X, let G be a mapping of X into itself that satisfies the inequality $|| G\alpha - G\beta || \leq || \alpha - \beta ||$. The class of contraction mapping is generally known to be non-expensive, and G is appropriately included in the class of all continuous mappings. For non-expansive mappings defined on a closed, bounded, and convex subset of a uniformly convex Banach space and in spaces with richer structure, Kirk (1965) separately demonstrated a fixed point theorem.
Many authors have considered various generalizations of non-expansive mappings. Particularly noteworthy are the works of Goebel (1969); Goebel and Zlotkiewicz (1971); Goebel, Kirk, and Shimi (1973); Massa and Roux (1978); Dotson (1972a and b); Emmanuele (1981); and Rhoades (1982). Kirk (1965, 1981, 1983) provides a thorough overview of fixed point theorems for non-expansive and related mappings.
However, certain mappings have a unique fixed point and meet constraints that are comparable to those of non-expansive mappings. However, these mappings cannot be thought of as extensions of non-expansive mappings. Recent instances of this type can be found in Rhoades (1978) and Gregus (1980). Inspired by a contractive condition of Hardy and Rogers (1973), we expand Gregus's (1980) solution to the situation of two mappings in this chapter.
Let M be a subset of X that is closed and convex. In conclusion, this author demonstrated the following outcome under the assumption that $y = z$ in Gregus's (1980) contractive condition.

1.1 Preliminaries
1.1.1 Banach space
Let X be a vector space over $\mathbb{R}$ or $\mathbb{C}$ and let || . || be a norm on X then (X, || . ||) is called a Banach space if every Cauchy sequence in X converges to a limit in X.

1.1.2 Non expansive mapping

Let C be a nonempty convex subset of a real Banach space E and ℝ be the set of real numbers. A mapping $T: C \rightarrow C$ is called non expansive if $\| Tx - Ty \| \leq \| x - y \|$ for all $x, y \in C$.

1.1.3 Fixed point

let T be a mapping from a se X into itself, i.e, T: X → X. A point x ∈ X is called a fixed point of the mapping T if T (x) = x.

1.1.4 Common fixed point

Let (X, d) be a metric space and let $T_1, T_2, T_3, \ldots, T_n$: X → X be mappings a point $x^* \epsilon$ X is called a common fixed point of these mapping if $T_1(x^*) = T_2(x^*) = T_3(x^*) = \ldots = T_n(x^*) = x^*$

1.1.5 Cauchy Sequence

Let (X, d) be a metric space. A sequence $\{x_n\} \subset X$ is known as Cauchy sequence if
∀ ε > 0, ∃ N ∈ ℕ such that $\| x_n - x_m \| < \varepsilon$ for all n, m ≥ N.

1.2 Theorem

Let G be a mapping of M by itself, so resolving the inequality

$$|| G\alpha - G\beta || \leq x. || \alpha - \beta || + y. \{|| G\alpha - \alpha || + || G\beta - \beta ||\} \quad (1)$$

for all α, β in M, where $0 < x < 1, y > 0$ and $x + 2y = 1$. Then G has a unique fixed point. The following theorem is now proven.

1.3 Main result

1.3.1 Theorem

Let F and G be mappings of M into themselves satisfying the inequality

$$|| F\alpha - G\beta || \leq x. || \alpha - \beta || + y. \{|| F\alpha - \alpha || + || G\beta - \beta ||\} \quad (2)$$

$$+ z. \{|| F\alpha - \beta || + || G\beta - \alpha ||\}$$

for all α, β in M, where $0 < x < 1, y > 0$ and $x + 2y + 2z = 1$ and $(1 - y).z < xy$. If

$$|| G\alpha - \alpha || \leq || F\alpha - \alpha || \quad (3)$$

∀ α in M, then F and G have a unique common fixed point w in M. Moreover, w is the unique fixed point of F and G.

Proof:

Let α be an arbitrary point in M. From (2), we deduce that

$$|| FG\alpha - G\alpha || \leq x. || G\alpha - \alpha || + y. \{|| FG\alpha - G\alpha || + || G\alpha - \alpha ||\}$$

$$+ z. \{|| FG\alpha - G\alpha || + || G\alpha - \alpha ||\} + t || F\alpha - G\alpha ||$$

Combining like terms

$$|| FG\alpha - G\alpha || \leq (x + y + z) || G\alpha - \alpha || + (y + z) || FG\alpha - G\alpha || + || F\alpha - G\alpha ||$$

Rewriting, we get

$$(1 - y - z) || FG\alpha - G\alpha || \leq (x + y + z) || G\alpha - \alpha || + || F\alpha - G\alpha ||$$

Assuming from condition 3

$$|| F\alpha - G\alpha || \leq || F\alpha - \alpha || + || G\alpha - \alpha || \leq 2 || F\alpha - \alpha ||$$

which implies that

$$|| FG\alpha - G\alpha || \leq \frac{x+y+z}{1-y-z} . || G\alpha - \alpha || + \frac{t}{1-y-z} .2 \; || F\alpha - \alpha || \quad (4)$$

$$|| FG\alpha - G\alpha || \leq A \; || G\alpha - \alpha || + B \; || F\alpha - \alpha ||$$

Where A = $\frac{x+y+z}{1-y-z}$, B = $\frac{2t}{1-y-z}$

Similarly, we have

$$\| GF\alpha - F\alpha \| \leq \| F\alpha - \alpha \|. \quad (5)$$

Since 5 holds ∀ α in M, we deduce that

$$\| FGF\alpha - FG\alpha \| \leq \| GF\alpha - F\alpha \|,$$

Which implies, by (3) and (5), that

$$\| GGF\alpha - GF\alpha \| \leq \| FGF\alpha - GF\alpha \| \leq \| F\alpha - \alpha \|. \quad (6)$$

We now define the point γ by

$$\gamma = \frac{1}{2}GF\alpha + \frac{1}{2}GGF\alpha$$

Then, it follows, from the above inequality, that

$$2 \| GF\alpha - \gamma \| = 2 \| GGF\alpha - \gamma \| = \| GGF\alpha - GF\alpha \| \leq \| F\alpha - \alpha \|. \quad (7)$$

Since M is convex, $\gamma \in M$ and using above steps and added the term t $\| F\alpha - G\alpha \|$ we have that

$$2 \| F\gamma - \gamma \| = \| 2F\gamma - (GF\alpha + GGF\alpha) \| = \| F\gamma - GF\alpha \| + \| F\gamma - GGF\alpha \| \quad (8)$$

Apply the contractive inequality to each term

$$\| F\gamma - GF\alpha \| \leq x. \| \gamma - F\alpha \| + y. \{\| F\gamma - \gamma \| + \| F\alpha - \alpha \|\}$$

$$+ z. \{\| F\gamma - \gamma \| + \| F\alpha - \gamma \| + \| GF\alpha - \gamma \|\} + \text{t} \| F\gamma - GF\gamma \|$$

$$\| F\gamma - GGF\alpha \| \leq x. \| \gamma - F\alpha \| + y. \{\| F\gamma - \gamma \| + \| F\alpha - \alpha \|\}$$

$$+ z. \{\| F\gamma - \gamma \| + \| GF\alpha - \gamma \| + \| GGF\alpha - \gamma \|\} + \text{t} \| F\gamma - GF\gamma \|$$

Combining these

$$2\| F\gamma - \gamma \| \leq x. \| F\alpha - \gamma \| + \| \gamma - GF\alpha \| + 2y. \{\| F\gamma - \gamma \| + \| F\alpha - \alpha \|\}$$

$$+ z. \{2 \| F\gamma - \gamma \| + \| F\alpha - \gamma \| + 3 \| F\alpha - \alpha \|\} + 2\text{t} \| F\gamma - G\gamma \|$$

Simplify using the bounds:

$$\| F\alpha - \gamma \| \leq \frac{1}{2} \| F\alpha - \alpha \|, \| GF\alpha - \gamma \| \leq \frac{1}{2} \| F\alpha - \alpha \|, \| GGF\alpha - \gamma \| \leq \frac{1}{2} \| F\alpha - \alpha \| \quad (9)$$

$$2\{\| F\gamma - \gamma \| \leq x. \{2. \frac{1}{2}. \| F\alpha - \alpha \|\} + 2y. \{\| F\gamma - \gamma \| + \| F\alpha - \alpha \|\} + z. \{2 \| F\gamma - \gamma \| + \frac{1}{2} \| F\alpha - \alpha \| + \frac{3}{2}. \| F\alpha - \alpha \|\} + 2\text{t} \{\| F\gamma - G\gamma \|$$

$$2\{\| F\gamma - \gamma \| \leq \text{x} \| F\alpha - \alpha \| + 2\text{y} \{\| F\gamma - \gamma \| + \| F\alpha - \alpha \|\} + \text{z} \{2 \| F\gamma - \gamma \| + 2 \| F\alpha - \alpha \| + 2\text{t} \| F\gamma - G\gamma \|$$

$$\| F\gamma - G\gamma \| \leq \| F\gamma - \gamma \| + \| G\gamma - \gamma \| \leq 2 \| F\gamma - \gamma \| + \| F\alpha - \alpha \|$$

$$2 \| F\gamma - \gamma \| \leq x. (2 - y). \| F\alpha - \alpha \| + 2y. \{\| F\alpha - \alpha \| + \| F\gamma - \gamma \|\} + z. \{2 \| F\gamma - \gamma \| + (3 - y). \| F\alpha - \alpha \|\}.$$

$$2 \| F\gamma - \gamma \| \leq (\text{x} + 2\text{y} + 2\text{z} + 2\text{t}) \| F\alpha - \alpha \| + (2\text{y} + 2\text{z} + 4\text{t}) \| F\gamma - \gamma \|$$

$$(2\text{-}2\text{y}\text{-}2\text{z}\text{-}4\text{t}) \| F\gamma - \gamma \| \leq (\text{x} + 2\text{y} + 2\text{z} + 2\text{t}) \| F\alpha - \alpha \|$$

Consequently, we have that

$$\| F\gamma - \gamma \| \leq \delta . \| F\alpha - \alpha \|, \qquad (10)$$

Where

$$\delta = \frac{x + 2y + 2z + 2t}{2 - 2y - 2z - 4t} \; x + 2y + 2z + 2t < 1 \Rightarrow \delta < 1$$

It follows that $0 < \delta < 1$ based on the assumptions made about the constants x, y, z *and* t. Specifically that x + 2y + 2z + 2t <1 We claim that

$$h = inf\{\| F\alpha - \alpha \| : \alpha \in M\} = 0,$$

suppose for the contradiction that h > 0 then for any $0 < \varepsilon < \frac{(1-\delta).h}{\delta}$, there exists a point $\bar{\alpha} \in M$ such that $\| F\bar{\alpha} - \bar{\alpha} \| \leq (h+\epsilon)$ then from above inequality we have

$$h \leq \| F\gamma - \gamma \| \leq \delta \| F\bar{\alpha} - \bar{\alpha} \| \leq \delta (h + \varepsilon) < h,$$

which is a contradiction . therefore the assumption h > 0 must be false and we conclude a Thus $h = 0$ and the sets

$$\mathbb{H}_n = \{\alpha \in M : \{\| F\alpha - \alpha \| < \frac{1}{n}\}\}$$

are non-empty for any $n = 1, 2, \dots$;

Now we have,

$$(11)\ \mathbb{H}_1 \supseteq \mathbb{H}_2 \supseteq \dots \supseteq \mathbb{H}_n \supseteq \dots$$

Let $\overline{\mathbb{H}}_n$ be the closure of $\mathbb{H}_n$. We now show that

(12) diam $\overline{\mathbb{H}}_n \leq \frac{(3-x)}{2yn}$ for any $n = 1, 2, \dots$. Indeed, we obtain on using (2)withadded term t $\| F\alpha - G\alpha \|$.

$$\| \alpha - \beta \| \leq \| F\alpha - \alpha \| + \| F\alpha - \beta \|$$

$$\leq \| F\alpha - \alpha \| + \| G\beta - \beta \| + \| F\alpha - G\beta \|$$

$\| F\alpha - G\beta \| \leq$ x $\| \alpha - \beta \|$ + y $\| F\alpha - \alpha \| + \| G\beta - \beta \|$ + z($\| F\alpha - \alpha \| + \| \alpha - \beta \| + \| G\beta - \beta \| + \| \alpha - \beta \| + t \| F\alpha - G\alpha \|$

We know $\| F\alpha - \alpha \| < \frac{1}{n}$, $\| G\beta - \beta \| \leq \| F\beta - \beta \| < \frac{1}{n}$ and $| F\alpha - G\alpha \|$

$\leq \| F\alpha - \alpha \| + \| G\alpha - \alpha \| \leq \frac{2}{n}$ so, $\leq \frac{2}{n} + x . \| \alpha - \beta \| + y . \{\| F\alpha - \alpha \| + \| G\beta - \beta \|\} + z . \{\| F\alpha - \alpha \| + \| \alpha - \beta \| + \| G\beta - \beta \| + \| \alpha - \beta \|\}$

$$\leq \frac{2}{n} + (x + 2z) . \| \alpha - \beta \| + \frac{(2y+2z)}{n} + \frac{2t}{n}$$

$$\| \alpha - \beta \| = \frac{(3-x)}{n} + (1 - 2y) . \| \alpha - \beta \|$$

By equation (3) $\| G\beta - \beta \| \leq \| F\beta - \beta \| \leq \frac{1}{n}$. By above inequality (12)
$diam\ \mathbb{H}_n = diam\ \overline{\mathbb{H}}_n$ and clearly it follows from (11) that

$$\overline{\mathbb{H}}_1 \supseteq \overline{\mathbb{H}}_2 \supseteq \dots \supseteq \overline{\mathbb{H}}_n \supseteq \dots$$

The series $diam\ \overline{\mathbb{H}}_n$ converges to zero as $n \to \infty$ by (12), indicating that $\{\overline{\mathbb{H}}_n\}$ is a decreasing sequence of non-empty subsets of M. Cantor's intersection theorem states that since X and M are complete, there is a point w in M such that

$$w \in \bigcap_{n=1}^{\infty} \overline{\mathbb{H}}_n.$$

Accordingly, $\| Fw - w \| \leq \frac{1}{n}$ for any $n = 1, 2, \ldots \ldots$, and so $Fw = w$, By using (3), we have $Gw = w$. Then, w is a fixed point that both F and G share. Assume that w' is an additional fixed point of F. With (2) applied to $\alpha = w$ and $\beta = w'$ we obtain that

$$\| w' - w \| = \| Fw' - Gw \| \leq x . \| w' - w \| + z . \{\| w' - w \| + \| w - w' \|\} + t \| Fw - Gw \|$$

$$= (x + 2z) . \| w' - w \|.$$

This implies that $w' = w$ since $x + 2z < 1$.

(1 -x- 2z) $\| w - w' \| \leq 0 = \| w - w' \| = 0 =$ w = w'

Therefore w is the unique fixed point of F and similarly it is shown that w is the unique fixed point of G. This completes the proof.

Remark

Theorem 5.3.1 becomes theorem 1 if $F = G$ and $z = 0$ are assumed.

The following outcome is obtained by enunciating theorem 5.3.1 for certain iterates of F and G.

1.4 Theorem

Let F and G be self maps on a non empty closed convex subset M of a Banach space $(X, \| . \|)$. Assume there exist constant $x, y, z, t \geq 0$ satisfying $x + 2y + 2z + 2t < 1$, and integers $u, v \in N$ such that for all $\alpha, \beta \in M$, the inequality holds:

$$\| F^u\alpha - G^v\beta \| \leq x \| \alpha - \beta \| + y (\| F^u\alpha - \alpha \| + \| G^v\beta - \beta \|) + z (\| F^u\alpha - \beta \| + \| G^v\beta - \alpha \|) + t \| F^u\alpha - G^v\alpha \|.$$

Assume also that:

$$\| G^v\beta - \alpha \| \leq \| F^u\alpha - \alpha \| \ \forall \alpha \in M.$$

Then F and G have a unique common fixed point $w \in M$.

Proof:

Let

$$\tilde{F}(\alpha) := F^u\alpha, \tilde{G}(\alpha) := G^v\alpha.$$

Then the contractive inequality becomes

$$\| \tilde{F}(\alpha) - \tilde{G}(\beta) \| \leq x \| \alpha - \beta \| + y (\| \tilde{F}(\alpha) - \alpha \| + \| \tilde{G}(\beta) - \beta \|) + z (\| \tilde{F}(\alpha) - \beta \| + \| \tilde{G}(\beta) - \alpha \|) + t \| \tilde{F}(\alpha) - \tilde{G}(\alpha) \|.$$

Also, the assumption becomes

$$\| \tilde{G}(\beta) - \alpha \| \leq \| \tilde{F}(\alpha) - \alpha \| \ \ \forall \alpha \in M.$$

Since all the assumptions of the earlier theorem (which included the term $t \| \tilde{F}(\alpha) - \tilde{G}(\alpha) \|$) are satisfied:
M is closed and convex in a Banach space. The inequality is satisfied with constants x, y, z, t such that $x + 2y + 2z + 2t < 1$,
The dominance condition $\| \tilde{G}(\beta) - \alpha \| \leq \| \tilde{F}(\alpha) - \alpha \|$ holds;

we conclude from that theorem that there exists a unique point $w \in M$ such that:

$$\tilde{F}(w) = w = \tilde{G}(w).$$

So,

$$F^u(w) = w = G^v(w)$$

We will prove that w is also a fixed point of F and G
From $F^u(w) = w$, apply F repeatedly:

$$F^u(w) = F^{u-1}(F(w)) = w \implies F(w) = F^u(w) = w.$$

Similarly, $G(w) = G^v(w) = w$.
So, w is a fixed point of F and G.
For uniqueness of fixed point Suppose $w' \in M$ is another point such that $F(w') = w', G(w') = w'$.
Then:

$$F^u(w') = w', G^v(w) = w'.$$

Apply the contractive inequality with $\alpha = w, \beta = w'$

$$|| w - w' || = || F^u(w) - G^v(w') || \leq x || w - w' || + y(0+0) + z(|| w - w' || + || w' - w ||) + t \mid \mid F^u(w) - G^v(w) ||.$$

But since w and w' are fixed points

$$F^u(w) = G^v(w) = w, F^u(w') = G^v(w') = w',$$

so $|| F^u(w) - G^v(w) || = 0$, and the inequality becomes

$$|| w - w' || \leq (x + 2z)(|| w - w' ||.$$

Since $x + 2z < 1$ (as implied by $x + 2y + 2z + 2t < 1$), it follows that

$$(1 - x - 2z) || w - w' || \leq 0 \implies (|| w - w' || = 0 \implies w = w'.$$

So the fixed point is unique. Under the given assumptions, the mappings F and G have a unique common fixed point $w \in M$, and $F(w) = w = G(w)$. The following example shows the stronger generality of theorem 5.3.3 over theorem 5.3.2.

Example
Let X be the Banach space of reals with Euclidean norm and $M = [0,2]$. We define F and G by putting $F\alpha = 0 \; if \; 0 \leq \alpha < 1, F\alpha = \frac{3}{5} \; if \; 1 \leq \alpha \leq 2, G\alpha = 0 \; if \; 0 \leq \alpha < 2 \; and \; G\alpha = \frac{9}{5}$ Then the condition (2) of theorem 5.2.1does not hold, otherwise, we should have for $\alpha = 1$ and $\beta = 2$

$$\frac{6}{5} = || F_1 - G_2 || \leq x.|| 2 - 1 || + y.\{|| 1 - \frac{3}{5} || + || 2 - \frac{9}{5} ||\} + z.\{|| \frac{9}{5} - 1 || + || 2 - \frac{3}{5} ||\}$$

$$= x + \frac{3y}{5} + \frac{11z}{5}$$

$$= 1 - 2y - 2z + \frac{3y}{5} + \frac{11z}{5}$$

Which implies $\frac{1}{5} + \frac{7y}{5} \leq \frac{z}{5}, i.e, 1 + 7y \leq z$, a contradiction. However, the conditions of theorem 5.3.3 are trivially satisfied for $u = v = 2$ since $F2\alpha = G2\alpha = 0$ for all α in M.
Although the contradictive condition used in this chapter is more general than (2), we explicitly note that the results for $F = G$ are not comparable to the results where the additional assumptions on the coefficients and the uniform convexity of X neither imply nor are implied by the assumptions of theorem 5.3.2.

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