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An Integral Operator with a Complex Parameter Defined by Subordination Principle

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Abstract
In this paper, two new subclasses $\mathfrak{H}(\mathrm{A}, \mathrm{B}, \alpha, \gamma, \tau, \sigma)$ and $\mathfrak{M}(\mathrm{A}, \mathrm{B}, \alpha, \gamma, \tau)$ of univalent functions defined in the unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$ are introduced by applying subordination principle. Analysis of the geometric properties of these new classes with emphasis on coefficient inequality and neigbhourhood property were carried out.

Keywords: Analytic Functions, Subordination, Neigbhourhood Property, Coefficient Inequalities, Univalent Functions

Introduction
Let f be the class of functions $f(z)$ defined by

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad (1)$$

which are analytic in the unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$. Denote by S the subclass of A, consisting of functions which are analytic, univalent in the unit disk $\mathbb{U}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$.

Let $\mathbb{T}$ denote the subclass of S consisting of functions whose non-zero coefficients, from the second on, are negative. That is, an analytic and univalent function $f(z) \in \mathbb{T}$ if it can be expressed as

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k, a_k \geq 0 \quad (2)$$

A function $f(z) \in S$ of the form (1) is star-like in the unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$ if it maps a unit disk onto a star-like domain. A necessary and sufficient condition for a function $f(z)$to be star-like is that

$$\mathbb{R}\left(\frac{zf'(z)}{f(z)}\right) > 0, z \in \mathbb{U}.$$

The class of all star-like functions can be denoted by S*
An analytic function $f(z)$ of the form (1)is convex if it maps the unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$ conformally onto a convex domain. Equivalently, a function $f(z)$ is said to be convex if and only if it satisfies the following condition;

$$\mathbb{R}\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, z \in \mathbb{U}.$$

The class of all convex functions can be denoted by C^*.
Let $f(z)$ and $g(z)$ be analytic functions in the unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$, then $f(z)$ is subordinate to $g(z)$ in the unit disk $\mathbb{U}$ written as $f(z) \prec g(z)$, if there exists a function $\omega(z)$, analytic in the unit disk satisfying the conditions $w(0) = 0, |\omega(z)| < 1$, which is called a Schwartz function, such that $f(z) = \omega\big(g(z)\big)$. If the function g is univalent in $\mathbb{U}$, the $f(z) \prec g(z), z \in \mathbb{U} \Leftrightarrow f(0) = g(0)$ and $f(U) \subset g(U)$.

Gamma function is a generalization of the factorial function to real and complex numbers (except $\mathbb{Z}^-$) and it is defined by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \ (\mathbb{R}(z) > 0)$$

Beta function can be referred to as the Euler integral of the first kind, defined by

$$\mathrm{B(x,y)} = \int_0^1 t^{x-1}(1-t)^{y-1}\, dt \ (x, y > 0)$$

For any function $f(z) \in \mathbb{T}$ and $\delta \geq 0, \delta$ −neighborhood of f is defined by

$$\beth_\delta(f) = \{g(z) = z - \sum_{k=2}^\infty b_k z^k \in \mathbb{T}\text{: } \sum_{k=2}^\infty k|a_k - b_k| \leq \delta\}$$

Notably, for the identity function $e(z) = z$,

$$\beth_\delta(e) = \{g(z) = z - \sum_{k=2}^\infty b_k z^k \in \mathbb{T}\text{: } \sum_{k=2}^\infty k|b_k| \leq \delta\}$$

Atshan *et al* [2], defined and study the integral operator $I(z)$, represented by

$$I(z) = Q_\gamma^\tau = \binom{\tau+\gamma}{\gamma} \frac{\tau}{z^\gamma} \int_0^z t^{\gamma-1} \left(1 - \frac{t}{z}\right)^{\tau-1} f(t) dt,$$

$(\tau > 0, \gamma > -1, z \in \mathbb{U})$.
and it can be easily verified that

$$I(z) = Q_\gamma^\tau = z - \sum_{k=2}^\infty \frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)} a_k z^k \quad (3)$$

Researchers in the geometric function theory have introduced several subclasses of analytic univalent functions defined in the unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$ and studied their geometric properties, ([5], [6], [7]). Integral operators play a vital role in geometric function theory and this has attracted the attention lot of researchers in this line, to further introduce new classes of univalent functions defined by subordination – based integral operator within the unit disk $\mathbb{U}$ and as well investigate their geometric properties. For further study, see ([1], [3], [8], [9]).
In light of this progression, the current study introduces an integral operator which extends the work of Atshan *et al* [2] by incorporating additional complex parameters.

Lemma 1 [2]

Let $Q(A, B, \alpha, n)$ consists of all analytic functions m in $\mathbb{U}$ for which $m(0) = 1$ and

$$m(z) = \frac{1+[B+\alpha((1-\alpha)+(A-B)]z}{1+Bz}$$

$$z \in \mathbb{U}, -1 \leq B < A \leq 1, 0 < \alpha \leq 1.$$

Definition 1: A function $f \in \mathbb{T}$ of the form (2) belongs to the class $\mathfrak{H}(A, B, \alpha, \gamma, \tau, \sigma)$ if

$$1 + \frac{1}{\sigma}\left\{\frac{zI'(z)}{I(z)} - 1\right\} \prec m(z) \quad (4)$$

$$\sigma \in \{\mathbb{C}\backslash 0\}, z \in \mathbb{U}, -1 \leq B < A \leq 1, 0 < \alpha \leq 1, \tau > 0, \gamma > -1.$$

Remark 1:

when $\sigma = 1$; $1 + \frac{1}{\sigma}\left\{\frac{zI'(z)}{I(z)} - 1\right\} \prec m(z)\ reduces\ to \frac{zI'(z)}{I(z)} \prec m(z)$ [2]

when $b = 1, \alpha = 1$; $1 + \frac{1}{b}\left\{\frac{zI'(z)}{I(z)} - 1\right\} \prec m(z)$ reduces to $z\frac{f'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}$ [8]

Definition 2: A function $f \in \mathbb{T}$ of the form (2) belongs to the class $\mathfrak{M}(A, B, \alpha, \gamma, \tau)$ if

$$I'(z) \prec m(z)$$

$$z \in \mathbb{U}, -1 \leq B < A \leq 1, 0 < \alpha \leq 1, \tau > 0, \gamma > -1.$$

2. Main Results

Coefficient Inequality

Theorem 1: A function $f \in \mathbb{T}$ of the form (2) belongs to the class $\mathfrak{H}(A, B, \beta, \rho\, \mu, \sigma)$ if and only if

$$\sum_{k=2}^{\infty}\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\{\sigma\alpha\big((1-\alpha)+(A-B)\big)-(k-1)(B-1)\}a_k \leq \sigma\alpha\big((1-\alpha)+(A-B)\big) \quad (5)$$

$\sigma \in \{\mathbb{C}\backslash 0\}, z \in \mathbb{U}, -1 \leq B < A \leq 1, 0 < \alpha \leq 1, \tau > 0, \gamma > -1.$

Proof: Let $f \in \mathfrak{H}(A, B, \beta, \rho\, \mu, \sigma)$. Then

$$1 + \frac{1}{\sigma}\left\{\frac{zI'(z)}{I(z)} - 1\right\} \prec m(z)$$

By the definition of subordination and upon simplification of (3),

$$\frac{zI'(z)}{I(z)} = \sigma m(\omega(z)) + 1 - \sigma$$

$$\omega(z) = \frac{zI'(z)-I(z)}{I(z)[\sigma\{B+\alpha(1-\alpha)+(A-B)\}-B(\sigma-1)]-BzI'(z)} \quad (6)$$

It can also be seen from the definition of subordination that $|\omega(z)| < 1$, so (6) is written as

$$\omega(z)| = \left|\frac{zI'(z)-I(z)}{I(z)[\sigma\{B+\alpha(1-\alpha)+(A-B)\}-B(\sigma-1)]-BzI'(z)}\right| < 1 \quad (7)$$

Using (3) in (7), it yields

$$\left|\frac{-\sum_{k=2}^{\infty}\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}(k-1)a_k z^k}{\sigma\alpha((1-\alpha)+(A-B))z-\sum_{k=2}^{\infty}\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\{B+\sigma\alpha\big((1-\alpha)+(A-B)\big)-Bk\}a_k z^k}\right| < 1$$

Since $\mathbb{R}(\varphi) \leq |\varphi| < 1, \varphi \in \mathbb{C}$. Then

$$\mathbb{R}\left\{\frac{-\sum_{k=2}^{\infty}\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}(k-1)a_k}{\sigma\alpha((1-\alpha)+(A-B))-\sum_{k=2}^{\infty}\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\{B+\sigma\alpha\big((1-\alpha)+(A-B)\big)-Bk\}a_k}\right\} < 1$$

Taking the value of z on the real axis as $z \to 1$

$$\sum_{k=2}^{\infty}\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\{\sigma\alpha\big((1-\alpha)+(A-B)\big)-(K-1)(B-1)\}a_k \leq \sigma\alpha((1-\alpha)+(A-B))$$

Conversely, Suppose $f(z) = z - \sum_{k=2}^{\infty} a_k z^k$ and

$$\sum_{k=2}^{\infty}\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\{\sigma\alpha\big((1-\alpha)+(A-B)\big)-(K-1)(B-1)\}a_k \leq \sigma\alpha((1-\alpha)+(A-B))$$

holds true, and

$$1 + \frac{1}{\sigma}\left\{\frac{zI'(z)}{I(z)} - 1\right\} \prec m(z)$$

Using the principle of subordination,

$$\left\{\frac{zI'(z)}{I(z)}\right\} = \sigma m\big(\omega(z)\big) + 1 - \sigma \quad (8)$$

On expansion of equation (8) and making $\omega(z)$ the subject of the equation

$$\omega(z)=\left|\frac{-\sum_{k=2}^{\infty}\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}(k-1)a_kz^k}{\sigma\alpha((1-\alpha)+(A-B))z-\sum_{k=2}^{\infty}\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\{B+\sigma\alpha((1-\alpha)+(A-B))-Bk\}a_kz^k}\right|$$

There is need to show that $|\omega(z)|<1$ in such a way that

$$|-\sum_{k=2}^{\infty}\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}(k-1)a_kz^k|<|\sigma\alpha((1-\alpha)+(A-B))z-\sum_{k=2}^{\infty}\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\{B+\sigma\alpha\big((1-\alpha)+(A-B)\big)-Bk\}a_kz^k|$$

$$|-\sum_{k=2}^{\infty}\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}(k-1)a_kz^k|-\left|\sigma\alpha\big((1-\alpha)+(A-B)\big)z-\sum_{k=2}^{\infty}\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\{B+\sigma\alpha\big((1-\alpha)+(A-B)\big)-Bk\}a_kz^k\right|<0$$

$$|-\sum_{k=2}^{\infty}\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}(k-1)a_kz^k|-\left|\sigma\alpha\big((1-\alpha)+(A-B)\big)z-\sum_{k=2}^{\infty}\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\{B+\sigma\alpha\big((1-\alpha)+(A-B)\big)-Bk\}a_kz^k\right|\le$$

$$|-\sum_{k=2}^{\infty}\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}(k-1)a_kz^k-\sigma\alpha\big((1-\alpha)+(A-B)\big)z-\sum_{k=2}^{\infty}\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\{B+\sigma\alpha\big((1-\alpha)+(A-B)\big)-Bk\}a_kz^k|$$

Taking $z=1$;

$$|-\sum_{k=2}^{\infty}\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}(k-1)a_k|-\left|\sigma\alpha\big((1-\alpha)+(A-B)\big)-\sum_{k=2}^{\infty}\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\{B+\sigma\alpha\big((1-\alpha)+(A-B)\big)-Bk\}a_k\right|\le$$

$$\sum_{k=2}^{\infty}\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\{\sigma\alpha\big((1-\alpha)+(A-B)\big)-(k-1)(B-1)\}a_k-\sigma\alpha((1-\alpha)+(A-B))\le 0$$

Hence, $f(z)\in\mathfrak{H}(A,B,\alpha,\tau,\gamma,\sigma)$.

Remark 2

1. When $\sigma=1$,

$$\sum_{k=2}^{\infty}\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\{\sigma\alpha\big((1-\alpha)+(A-B)\big)-(k-1)(B-1)\}a_k-\sigma\alpha\big((1-\alpha)+(A-B)\big)\le 0 \text{ reduces to}$$

$$\sum_{k=n+1}^{\infty}\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]\{\alpha\big((1-\alpha)+(A-B)\big)-(k-1)(B-1)\}a_k-\alpha\big((1-\alpha)+(A-B)\big)\le 0$$ [2]

Inclusion Relations

Theorem 2: A function $f\in\mathbb{T}$ of the form (2) belongs to the class $\mathfrak{H}(A,B,\alpha,\tau,\gamma)$ if and only if

$$(1-B)\sum_{k=2}^{\infty}k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]a_k\le(A-B) \tag{9}$$

$$z\in\mathbb{U},-1\le B<A\le 1,0<\alpha\le 1,\tau>0,\gamma>-1.$$

Proof.

Let $f(z)\in\mathbb{T}$ belong to the class (A,B,α,γ,τ), then by the definition of class $\mathfrak{M}(A,B,\alpha,\gamma,\tau)$

$$I'(z)\prec\frac{1+Az}{1+Bz}$$

where

$$I(z)=z-\sum_{k=2}^{\infty}\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]a_kz^k$$

$$I'(z)=1-\sum_{k=2}^{\infty}k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]a_kz^{k-1}$$

Using the subordination principle,

$$1-\sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k z^{k-1}=\frac{1+A\omega(z)}{1+B\omega(z)}$$

$$1+A\omega(z)=(1-\sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k z^{k-1})(1+B\omega(z))$$

which implies that

$$|\omega(z)|=\left|\frac{\sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k z^{k-1}}{(A-B)+B\sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k z^{k-1}}\right|<1$$

$$\mathbb{R}\left\{\frac{\sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k z^{k-1}}{(A-B)+B\sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k z^{k-1}}\right\}<1$$

Using $|z|=r, 0<r<1$;

$$\sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k r^{k-1} \leq (A-B)+B\sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k r^{k-1}$$

$$\sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k r^{k} \leq (A-B)r+B\sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k r^{k}$$

If the value of r is chosen on the real axis as $r\to 1$, then

$$(1-B)\sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k r^{k} \leq (A-B)$$

It is important to show that

$$I'(z) \prec \frac{1+Az}{1+Bz}$$

i.e.

$$1-\sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k z^{k-1}=\frac{1+A\omega(z)}{1+B\omega(z)}$$

Since

$$|\omega(z)|=\left|\frac{\sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k z^{k-1}}{(A-B)+B\sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k z^{k-1}}\right|$$

We need to show that $|\omega(z)|<1$, which implies that

$$\left|\sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k z^{k-1}\right| < \left|(A-B)+B\sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k z^{k-1}\right|$$

$$\left|\sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k z^{k-1}\right| - \left|(A-B)+B\sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k z^{k-1}\right|<0$$

$$\left|\sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k z^{k}\right| - \left|(A-B)z+B\sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k z^{k}\right|$$

$$\leq \left|\sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k z^{k} - (A-B)z - B\sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k z^{k}\right|$$

$$\leq \sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k - (A-B) - B\sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k$$

$$\leq (1-B)\sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k - (A-B) \leq 0$$

Which gives that $|\omega(z)|<1$ by maximum modulus theorem. Hence, $f \in \mathfrak{M}(A,B,\alpha,\gamma,\tau)$

Theorem 3:
Let

$$\delta = \frac{A-B}{(1-B)k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]}$$

Then $\mathfrak{M}(A, B, \alpha, \gamma, \tau) \in ב_\delta(e)$

Proof.
If $f \in \mathfrak{M}(A, B, \alpha, \gamma, \tau)$ then it can be seen from (9) that

$$(1-B)\sum_{k=2}^{\infty} k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right] a_k \leq A-B$$

$$\sum_{k=2}^{\infty} k a_k \leq \frac{A-B}{(1-B)k\left[\frac{\Gamma(\gamma+k)\Gamma(\tau+\gamma+1)}{\Gamma(\tau+\gamma+k)\Gamma(\gamma+1)}\right]} = \delta$$

Which implies that $f \in ב_\delta$
Therefore, $\mathfrak{M}(A, B, \alpha, \gamma, \tau) \in ב_\delta$

Conclusion
Two new subordination-based classes of analytic functions defined in the unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$, by incorporating a complex parameter into the framework of Atshan *et al* [2], were introduced. We established coefficient inequality for the functions in this class and investigated it's neigbhourhood properties, the results obtained generalized the original work of Atshan *et al* [2].

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