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A Note on Stochastic Pendulum Equation with Random Forcing

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Abstract
This study investigates the dynamics of the stochastic pendulum differential equation under the influence of white noise. The equation serves as a key mathematical model to describe physical systems affected by random disturbances. The main objective is to analyze the system's behavior in response to stochastic perturbations and to examine the impact of key parameters such as damping, noise intensity, and gravity on the stability and motion of the pendulum. To address the stochastic component of the equation, the Wronskian determinant method is employed. This approach enables an analytical solution and provides reliable results under uncertainty. The findings highlight the significant role of random noise in altering the dynamic behavior of the system, offering deeper insights into stochastic systems and nonlinear dynamics.

Keywords: Stochastic Pendulum Equation, White Noise, Wronskian Determinant

1. Introduction

In many physical and engineering systems, the behavior of dynamic systems is influenced by certain factors that act randomly and cannot be predicted. One of these factors is white noise, which is usually considered a random process [1].

White noise can have significant effects on the behavior of dynamic systems. One example is the pendulum. Since pendulums are important oscillating systems, white noise can affect them and cause unexpected changes in their behavior. These changes can lead to chaotic motion and alterations in their oscillations [2].

When the stochastic differential equations of a pendulum are influenced by white noise, the modeling and simulation of the pendulum can impact the stability of the system's oscillations in unpredictable ways [3].

The pendulum is one of the simplest yet most complex models in physics. In the real world, this system is affected by noise and random disturbances that may change its characteristics. Modeling the motion of a pendulum under the influence of white noise, treated as a random process, can alter its behavior [4].

Studying nonlinear dynamics in physical systems under white noise is an important area in the analysis of stochastic differential equations [5].

The pendulum, as a classical model in dynamics, can be randomly influenced by white noise, which can make the system's nonlinear behavior chaotic. These nonlinear behaviors and their stochastic differential equations are not only important but also widely used in modeling real-world phenomena in engineering, biology, and finance [6].

Many studies have explored stochastic differential equations and the effect of white noise on dynamic systems. Some of the key research works in this area are summarized below:

Tepljakov, Aleksei in, [10] the modeling of stochastic dynamic systems and methods for solving stochastic differential equations are introduced.

Bain,leeJ.and Max Engelhardt in, [14] discusses numerical techniques and the use of the Jacobian matrix in solving stochastic differential equations.

Zafar,Ana,and Randa Herzallah, [15] the authors investigate stochastic processes and the effects of white noise, especially in the fields of physics and chemistry.

Szuminski, Wojciech, [16] analyzes chaotic behaviors in nonlinear systems, focusing particularly on pendulums, and discusses the significant effects on amplitude and frequency.

More recent studies such as, [17] and [18] show that white noise can lead to chaotic behavior and phase transitions in stochastic pendulum systems. These studies focus on modeling and simulating the nonlinear dynamics of pendulums affected by white noise and introduce new algorithms for analyzing such systems.

The nonlinear dynamics of pendulum systems under random disturbances have received considerable attention. AlhejailiWeaam,Alvaro in, [19], the stability of a nonlinear pendulum with a moving suspension point under white noise is examined. Using Poincare mapping and an averaged Hamiltonian system, the study analyzes the stability conditions.

The stochastic duffing equation is a significant model in analyzing nonlinear oscillatory systems influenced by white noise, exhibiting complex dynamic behavior. Understanding how its parameters affect system dynamics under stochastic influences is crucial. Despite advances in numerical approaches, the analytical treatment of the stochastic component has received limited attention. This study addresses this gap by applying the Wronskian determinant method to analyze the stochastic part of the equation and investigate the parametric effects theoretically.

This paper analyzed the stochastic differential equation of the pendulum under the influence of white noise. The simulation results demonstrate that stochastic noise can have significant effects on the oscillatory behavior of the pendulum, causing unexpected changes in the system's motion. This research can contribute to improving the design of mechanical systems affected by environmental stochastic noise.

The simulation results for different values of white noise demonstrate significant variations in the behavior of the pendulum. In particular, as the intensity of the white noise increases, the amplitude of the system's oscillations also increases, causing the pendulum's motion to shift from regular and periodic movements toward unpredictable and chaotic behavior. These changes in oscillation amplitude and the nature of the pendulum's motion are clearly influenced by the noise intensity.

By applying the Wronskian determinant method, the stochastic part of the Duffing equation was analytical solved, revealing the behavior under white noise.

This study provides a concise analytical framework for examining the stochastic Duffing oscillator, offering a deeper understanding of its parameter-driven dynamics without relying on purely numerical methods.

2. Model description

The equation:

$$\varphi^{(n)}(t) + \alpha^2\varphi(t) = \sigma w'(t)$$

It is a linear stochastic differential equation (SDE) of order n, driven by a white noise term. This equation combines the structure of classical deterministic differential equations with the randomness introduced through stochastic processes [19].

2.1. $\varphi(t)$

This represents the unknown stochastic processes (solution function) we aim to determine. It evolves over time and is influenced by both deterministic dynamics and random fluctuations.

2.2. $\varphi^{(n)}(t)$

This denotes the n-th derivative of $\varphi(t)$ with respect to time. The order n determines whether the equation is first- order, second-order, etc. For example, if$n = 2$, the equation describes a stochastic oscillator or a noisy vibrating system.

2.3. $\alpha^2\varphi(t)$

The term α^2 is a positive constant related to the system's frequency or stiffness. The presence of $\varphi(t)$ ensures the system has a restoring force, similar to a spring in mechanical systems.

2.4. $\sigma w'(t)$

$w'(t)$ Represents white noise, which is the formal derivative of a wiener process (Brownian motion). It models unpredictable, high- frequency disturbances.σ Is a positive constant that scales the intensity of the noise [20].

2.5. Interpretation

This equation models a system where the state $\varphi(t)$ is not only affected by deterministic rules (like oscillation) but also experiences random shocks over time. Such models are widely used in physics, engineering, finance, and biological systems.

3. Mean results

To model the motion of the pendulum under the influence of white noise, the corresponding stochastic differential equation is formulated as follows. Subsequently, the equation is analyzed and solved based on the Wronskian determinant method.

$\varphi''(t) + \alpha^2\varphi(t) = \sigma w'(t) = \int_0^t N_s\, ds$

$\lambda^2 + \alpha^2 = 0 \Rightarrow \lambda^2 = -\alpha^2, \lambda_{1,2} = \pm\alpha i\ \varphi_n(t) = A\cos\alpha t + B\sin\alpha t\ W_r(t) = \begin{vmatrix} \cos\alpha t & \sin\alpha t \\ -\alpha\sin\alpha t & \alpha\cos\alpha t \end{vmatrix} = \alpha\cos^2\alpha t + \alpha\sin^2\alpha t \Rightarrow$

$W_r(t) = \alpha$

$\varphi_p(t) = -u_1\int \frac{u_2.g}{W_r}dt + u_2\int \frac{u_1.g}{W_r}dt$

$\varphi_p(t) = -\cos\alpha t\int \frac{\sin\alpha t.\sigma w'}{\alpha}dt + \sin\alpha t\int \frac{\cos\alpha t.\sigma w'}{\alpha}dt$

$\varphi_p(t) = \frac{-\sigma\cos\alpha t}{\alpha}\int \sin\alpha t\frac{dw}{dt}dt + \frac{\sigma\sin\alpha t}{\alpha}\int \cos\alpha t\frac{dw}{dt}dt$

$\varphi_p(t) = \frac{-\sigma\cos\alpha t}{\alpha}\int \sin\alpha\, sdw_s + \frac{\sigma\sin\alpha t}{\alpha}\int \cos\alpha\, sdw_s$

$\varphi(t) = \varphi_n(t) + \varphi_p(t)$

$\varphi(t) = A\cos\alpha t + B\sin\alpha t + \frac{\sigma\sin\alpha t}{\alpha}\int \cos\alpha\, sdw_s - \frac{\sigma\cos\alpha t}{\alpha}\int \sin\alpha\, sdw_s$

$\varphi(t) = \sin\alpha t(B + \frac{\sigma}{\alpha}\int_0^t \cos\alpha\, sdw_s) + \cos\alpha t(A - \frac{\sigma}{\alpha}\int_0^t \sin\alpha\, sdw_s$

$if: \varphi(0) = k_1\varphi'(0) = k_2$

$\varphi(t) = \cos\alpha t(A - \frac{\sigma}{\alpha}\int_0^t \sin\alpha\, sdw_s) + \sin\alpha t(B + \frac{\sigma}{\alpha}\int_0^t \cos\alpha\, sdw_s)$

$\varphi(t) = \cos\alpha t\left[A - \frac{\sigma}{\alpha}(w_t.\sin(\alpha t) - \alpha\int_0^t w_s.\cos(\alpha s)ds)\right] + \sin\alpha t\left[B + \frac{\sigma}{\alpha}(w_t.\cos(\alpha t) + \alpha\int w_s.\sin(\alpha s)ds)\right]$

$$v_1 = -\frac{\sigma}{\alpha}\int_0^t \sin\alpha s dw_s \qquad v_2 = \frac{\sigma}{\alpha}\int_0^t \cos\alpha s dw_s$$

$$E(v_1) = 0 \qquad , E(v_2) = 0$$

$$\text{var}(v_1) = E(v_1^2) - [E(v_1)]^2 \Rightarrow E(v_1^2) - 0 = E(v_1^2)$$

$$\text{var}(v_1) = E\left[(-\frac{\sigma}{\alpha}\int_0^t \sin\alpha s dw_s)^2\right] = \frac{\sigma^2}{\alpha^2}\int_0^t \sin^2\alpha s ds = \frac{\sigma^2}{\alpha^2}(\frac{t}{2} - \frac{\sin 2\alpha t}{4\alpha})$$

$$\text{var}(v_2) = E(v_2^2) - [E(v_2)]^2 \Rightarrow E(v_2^2) - 0 = E(v_2^2)$$

$$\text{var}(v_2) = E\left[(\frac{\sigma}{\alpha}\int_0^t \cos\alpha s dw_s)^2\right] = \frac{\sigma^2}{\alpha^2}\int_0^t \cos^2\alpha s ds = \frac{\sigma^2}{\alpha^2}(\frac{t}{2} + \sin 2\alpha t)$$

$$\text{cov}(v_1, v_2) = E[v_1.v_2] - E(v_1).E(v_2)$$

$$\text{cov}(v_1, v_2) = E\left[-\frac{\sigma}{\alpha}\int_0^t \sin\alpha s dw_s . \frac{\sigma}{\alpha}\int_0^t \cos\alpha s dw_s\right]$$

$$\text{cov}(v_1, v_2) = -\frac{\sigma^2}{\alpha^2}\int_0^t \sin\alpha s.\cos\alpha s ds = -\frac{\sigma^2}{\alpha^3}.\frac{\sin^2\alpha t}{2}$$

$E[\varphi(t)] = E\left[\cos\alpha t(A - \frac{\sigma}{\alpha}\int_0^t \sin\alpha\, sdw_s) + \sin\alpha t(B + \frac{\sigma}{\alpha}\int_0^t \cos\alpha\, sdw_s)\right]$

$E[\varphi(t)] = E[\varphi_n(t)] = A\cos\alpha t +$
$B\sin\alpha t\ var[\varphi(t)] = var[\varphi_p(t)] = E\left[(-\frac{\sigma\cos\alpha t}{\alpha}\int_0^t \sin\alpha\, sdw_s + \frac{\sigma\sin\alpha t}{\alpha}\int_0^t \cos\alpha\, sdw_s)^2\right] var[\varphi(t)] = \cos^2\alpha t\ var(v_1) + \sin^2\alpha t\ var(v_2) +$
$2\cos\alpha t\sin\alpha t.cov(v_1, v_2)$

$var[\varphi(t)] = \frac{\sigma^2(t)}{2\alpha^2} - \frac{\sigma^2(t)}{4\alpha^3}\sin 2\alpha t$

Example1: Second – Order Case ($n = 2$) let's take: $\alpha = 2, \sigma = 1$
Equation: $\varphi''(t) + 4\varphi(t) = w'(t)$ This is a stochastic harmonic oscillator.
Homogeneous solution: $\lambda^2 + 4 = 0 \Rightarrow \lambda = \pm 2i$

$\varphi h(t) = c_1\cos(2t) + c_2\sin(2t)$

Particular solution: Use Green's function G (t)

$G(t) = (\frac{\sigma}{\alpha})\int_0^t \sin[\alpha(t-s)]\, dw(s)$

$G(t) = (\frac{1}{2})\int_0^t \sin[2(t-s)]\, dw(s)$

$G(t) = (0{,}5)\int_0^t \sin[2(t-s)]\, dw(s)$

Full solution:

$\varphi(t) = \varphi h(t) + G(t)$

$\varphi(t) = c_1\cos(2t) + c_2\sin(2t) + (0{,}5)\int_0^t \sin[2(t-s)]\, dw(s)$

Mean and Variance:
$E[\varphi(t)] = c_1 \cos(2t) + c_2 \sin(2t)$ Stochastic integral has zero mean

Variance of the stochastic part:

$$Var[\varphi(t)] = \int_0^t \left[\left(\frac{1}{2}\right) \sin(2(t-s))\right]^2 ds$$
$$Var[\varphi(t)] = \left(\frac{1}{4}\right) \int_0^t \sin^2(2(t-s)ds \Rightarrow \int_0^t \frac{(1-\cos(4(t-s)))}{2} ds$$
$$Var[\varphi(t)] = \left(\frac{1}{8}\right)\left[t - \left(\frac{1}{4}\right)\sin(4t)\right]$$

Now, we want to plot the graphs of the Stochastic and Deterministic differential equation of the pendulum.

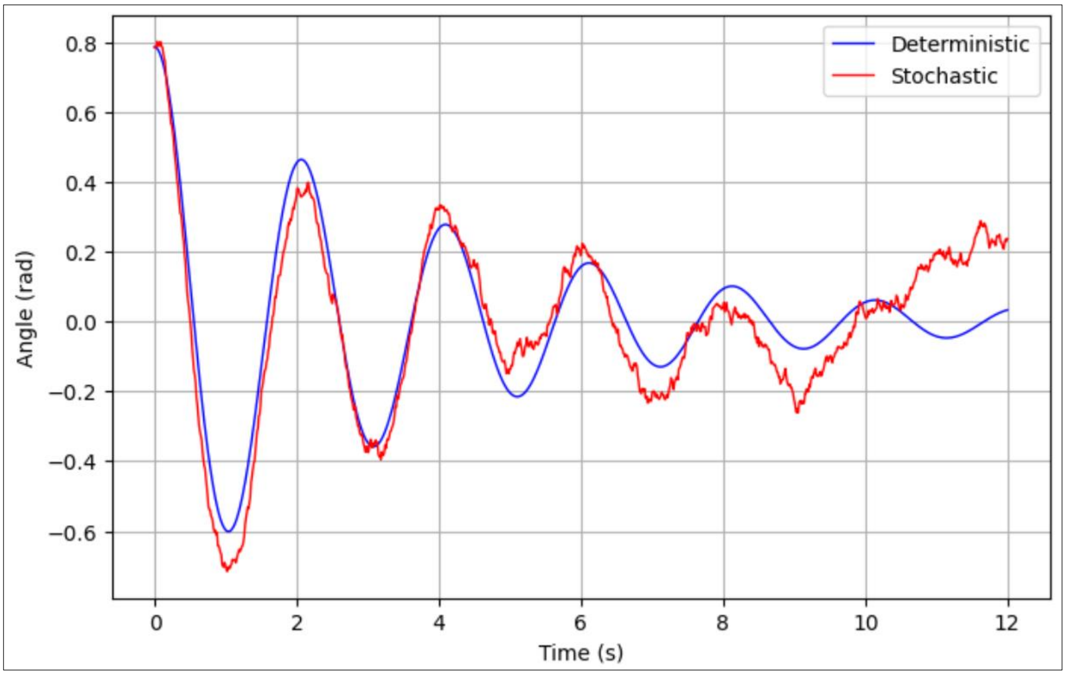

Fig 1: Pendulum Motion: Deterministic vs Stochastic

4. Conclusion
In this study, we explored the stochastic Duffing equation influenced by white noise, focusing on its dynamic behavior. By employing the Wronskian determinant method, the stochastic component was solved analytically, allowing a clear observation of how system parameters influence the response. This analytical approach not only highlights the underlying dynamics but also provides a valuable alternative to numerical simulations for understanding stochastic nonlinear systems.

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