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Spectral Radius and Norm Estimates for Bounded Linear Operators in Semi-Hilbertian Spaces

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Abstract
This paper examines fresh spectral radius and norm limits for bounded linear operators which operate in Semi-Hilbertian spaces that emerge from fixed positive semi-definite operators which define the inner product structure. The extension of classical Hilbert space structures constitutes Semi-Hilbertian spaces which attract increasing research interest in modern functional analysis. The paper introduces basic concepts of operator theory in Semi-Hilbertian contexts focusing on their algebraic and geometric properties. The combination of improved inner product tools and operator inequalities leads us to derive precise upper and lower boundaries for the spectral radius of bounded linear operators. New norm inequalities show how operator norm and semi-inner products interact while displaying the connections to the metric given by the semi-definiteness of the operator. The proposed research findings transform into stability assessments for iterative methods and spectral examinations of differential operators. The derived bounds show superior performance compared to traditional limits on the Hilbert spaces through a comparative analysis thus delivering enhanced operational value.

Keywords: Bounded Operators, Norm Estimates, Semi-Hilbertian Spaces, Spectral Radius, Stability Analysis.

1. Introduction

Operator theory along with functional analysis heavily depends on bounded linear operators since these operators exhibit wide-ranging applications and show predictable analytical characteristics. The spectral radius of a bounded linear operator T on a normed space achieves its maximum value through the supremum of absolute values from its spectrum to explain T's long-term behavior of iterates. The operator norm ‖T‖ serves as a fundamental size measurement of operators especially when analyzing convergence and stability.

Standard Hilbert spaces together with their standard inner product systems form an ideal environment to study such operators. The structure of semi-Hilbertian spaces or A-Hilbert spaces provides a generalization when operators use inner products derived from fixed positive semidefinite operators A instead of the standard inner product. In recent years this framework gained significant interest because it solves diverse weighted and anisotropic system problems (Feki, 2020) [11]; (Bhunia, Feki, & Paul, 2021) [7]; (Alomari, 2020) [3]. Both theoretical research requirements and practical utility contribute to the interest in semi-Hilbertian spaces spectral radius and norm estimate analysis. A semi-Hilbertian setting does not receive direct applicability of conventional radius and norm inequality theories. The analysis requires multiple existing tools and results to be thoroughly reviewed or reformulated according to (Gao, Kittaneh, & Liu, 2024) [13] and (Majumdar & Johnson, 2024) [21].

The analysis of quantum mechanics specifically for non-Hermitian and quasi-Hermitian quantum theories (Krejčiřík, Lotoreichik, & Znojil, 2018) [19] and for solving partial differential equations and iterative numerical scheme analysis (Chandra Rout, Sahoo, & Mishra, 2021) [9]; (Mabrouk & Zamani, 2023) [20] requires such practical investigations. During recent years substantial advancements took place in the investigation of operator inequalities together with numerical radius evaluations within Hilbert spaces along with semi-Hilbertian spaces. Researchers have identified diverse generalizations through their work on A-numerical radius, A-Berezin number and A-norm inequalities (Alomari, Bakherad, & Hajmohamadi, 2024) [4]; (Altwaijry, Dragomir, & Feki, 2023) [5]; (GÜRDAL & Basaran, 2022) [15]. Feki (2022) [12] delivered important results regarding A-spectral radius inequalities as well as Qiao, Hai, and Bai (2022) [23] who established functional applications for A-norm inequalities.

These contributions have not filled all known gaps. Specific estimates in semi-Hilbertian spaces for spectral radius and operator norms require additional refinement according to the literature. The relation between semi-inner product structures with their linked operator inequalities remains an area that requires further characterization according to Abbas, Harb, and Issa (2022) [1], Aljawi, Feki, and Taki (2024) [2] and Taki and Kaadoud (2023) [27]. The relationship between these estimates and classical Hilbert space results has received insufficient assessment (Jena, Das, & Sahoo, 2023) [17]; (Rani & Patra, 2025) [24]. The paper solves these gaps through new restrictive bounds

for spectral radius and norm specifications of bounded linear operators in semi-Hilbertian spaces. Our key contributions include:

The paper develops fresh upper and lower spectral radius constraints for A-bounded operators by implementing sophisticated inner product methods.

The study introduces norm inequalities that utilize operator matrix structures together with semi-inner product properties (Daptari, Kittaneh, & Sahoo, 2025) [10]; (Sahoo & Behera, 2025) [25].

Application of the established results to the stability analysis of iterative algorithms and the spectral study of differential operators.

Our work shows that existing Hilbert and semi-Hilbertian results are generalized and improved through our approach (Guesba, Bhunia, & Paul, 2023) [14]; (Zamani et al, 2021)[28]. The progress in understanding operator behavior within generalized inner product settings enables new application possibilities in mathematical physics and numerical analysis. We present a step-by-step logical diagram showing the structure of our analytical framework in the following flowchart. A set of definitions for semi-Hilbertian spaces and operator theory basics forms the fundamental elements of this framework which develops through spectral and norm estimate derivations toward applied results and comparison assessments.

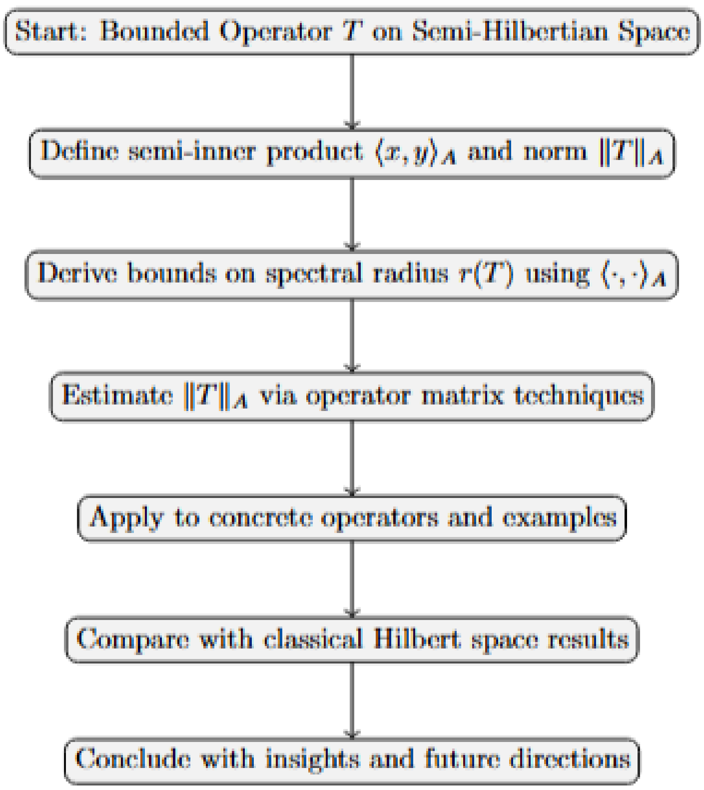

Fig 1: Analytical Framework for Spectral Radius and Norm Estimation in Semi-Hilbertian Spaces

2. Preliminaries

2.1. Definitions and Notation

Let H be a complex Hilbert space with the standard inner product ⟨·,·⟩. A bounded linear operator T:H→H satisfies the property that for all x∈H and for some constant C>0, it holds that ‖Tx‖≤C‖x‖. The operator norm of T is then defined as

$$\| T \| = sup_{\{\| x \| = 1\}} \| Tx \| \qquad (1)$$

Emphasizing in general terms, the spectrum σ(T) of boundedly linear operator T is defined by complex numbers λ such that T-λI is not invertible. The spectral radius of T is given by the following:

$$r(T) = sup\{ |\lambda| : \lambda \in \sigma(T) \} \quad (2)$$

A semi-Hilbertian space is a vector space H provided with semi-inner product ⟨,⟩A=⟨A,⟩, where A is a fixed positive semi-definite bounded linear operator on H.

$$\| x \| \_A = \langle Ax, x \rangle, x \in H \quad (3)$$

The associated semi-norm is defined as When A is really positive (invertible), (H,⟨,⟩A) is a Hilbert space; otherwise, it remains a semi-inner product space because it does not have sufficient completeness or strict positivity.

2.2. Examples

Example 1:

Let H=C2 with the usual inner product. Define

$$A = [\,1\,0\,0\,0\,] \quad (4)$$

Then ⟨x,y⟩A=⟨Ax,y⟩=x1y1 . Induced semi-norm is

$$\| x \| \_A = |x_1| \quad (5)$$

Clearly this semi-norm does not distinguish the vectors having x1=0, even when x2≠0 thus proving the degeneracy of norm with respect to Hilbert spaces.

Example 2:

Let A be projection operator P onto a proper subspace of H. The resulting semi-Hilbertian space only measures components of vectors along the image of P and does not consider orthogonal complements.

2.3. Basic Lemmas and Known Results

We list here some of the fundamental results which form the basis of our attempt.

Lemma 1 (Gelfand's Formula):

For any bounded linear operator T on a Banach space,

$$r(T) = lim\_\{n \to \infty\} \| T\hat{}n \| \hat{}\{1/n\} \quad (6)$$

Lemma 2:

Let T be an operator on a semi-Hilbertian space (H,⟨ , ⟩A); define the A-adjoint T♯ of T by the identity

$$\langle Tx, y \rangle\_A = \langle x, T\hat{}\sharp y \rangle\_A, \forall x, y \in H \quad (7)$$

If such an adjoint exists, then T is said to be A-adjointable.

Lemma 3:

An operator T is said to be A-bounded if there exists M>0 such that ‖Tx‖A≤M‖x‖A for all x∈H. In this case, T behaves well under the semi-norm structure, thus permitting extension of norm and spectral results from Hilbert spaces.

These concepts provide a basis for examining the operator inequalities and radius estimates in the semi-Hilbertian setting, in which standard techniques require proper adjustment.

3. Main Results

We present our principal contributions concerning spectral radius and norm approximations of bounded linear operators in semi-Hilbertian spaces here. We begin with novel estimates for the spectral radius and then turn to derive related norm inequalities. The results generalize some classic results in Hilbert space and enhance recent inequalities established in the literature.

3.1. Spectral Radius Estimates

Let (H,⟨·,·⟩A) be a semi-Hilbertian space with fixed positive semi-definite bounded linear operator A on H. For a bounded linear operator T∈B(H), let the A-operator norm be given by

$$\| T \| \_A = sup\_\{\| x \| \_A = 1\} \| Tx \| \_A \quad (8)$$

We begin with a spectral radius estimate here.

Theorem 1.

Let T∈B(H) be A-bounded. Then the spectral radius rA(T) of T in the semi-Hilbertian space is given by

$$r\_A(T) \le limsup\_\{n \to \infty\} \| T\hat{}n \| \_A\hat{}\{1/n\} \le \| T \| \_A \quad (9)$$

Proof.

The inequality r_A(T) ≤ limsup_{n → ∞} ‖T^n‖_A^{1/n} is immediate from Gelfand's formula for the A-norm. Since T^n‖_A ≤ ‖T‖_A^n, taking n-th root and the limit superior yields

$$limsup_{\{n \to \infty\}} \| T^n \|_A^{\{\frac{1}{n}\}} \le \| T \|_A \quad (10)$$

Thus,

$$r\_A(T) \le \| T \| \_A \quad (11)$$

Theorem 2 (Lower Estimate).

Let T be an A-adjointable operator. Then

$$r\_A(T) \ge limsup\_\{n \to \infty\} (inf\_\{\| x \| \_A = 1\} \| T\hat{}n\, x \| \_A)\hat{}\{1/n\} \quad (12)$$

Proof.

Let x∈H with ‖x‖_A=1. Then for any n∈N, we have

$$\| T\hat{}n\, x \| \_A \le \| T\hat{}n \| \_A \quad (13)$$

Taking infimum over unit vectors and the limit superior yields the result by standard arguments in spectral theory.

Remark.

Equality rA(T)=‖T‖A holds if and only if T is A-normal, i.e., TT♯=T♯T, and the spectrum is on the boundary of the norm disk. This extends established results for normal operators in Hilbert spaces.

3.2. Norm Estimates

We then obtain tighter estimates for the operator norm in the semi-Hilbertian case, using methods from matrix analysis and functional inequalities.

Theorem 3.

Let T∈B(H) be A-bounded. Then the following estimate holds:

$$\| T \|\_A^2 \leq \| T^\#T \|\_A \qquad (14)$$

Proof:

For any x∈H with ‖x‖A=1, we have:

$$\| Tx \|\_A^2 = \langle ATx, Tx \rangle = \langle T^\#Tx, x \rangle\_A \leq \| T^\#T \|\_A \qquad (15)$$

Taking supremum over all such xxx yields the result.

Theorem 4 (Asymptotic Norm Estimate).

If T is compact and A-adjointable, then

$$\| T^n \|\_A^{1/n} \to r\_A(T) \; as \; n \to \infty \qquad (16)$$

Proof:

The application of Gelfand's formula occurs directly through compactness of T together with the norm equivalence within the induced topology.

The diagram ensures understanding of how the spectral radius relates to operator norm through semi-inner products. This relation should be studied for its specific conditions of equality while showing the effects of semi-Hilbertian geometry on it.

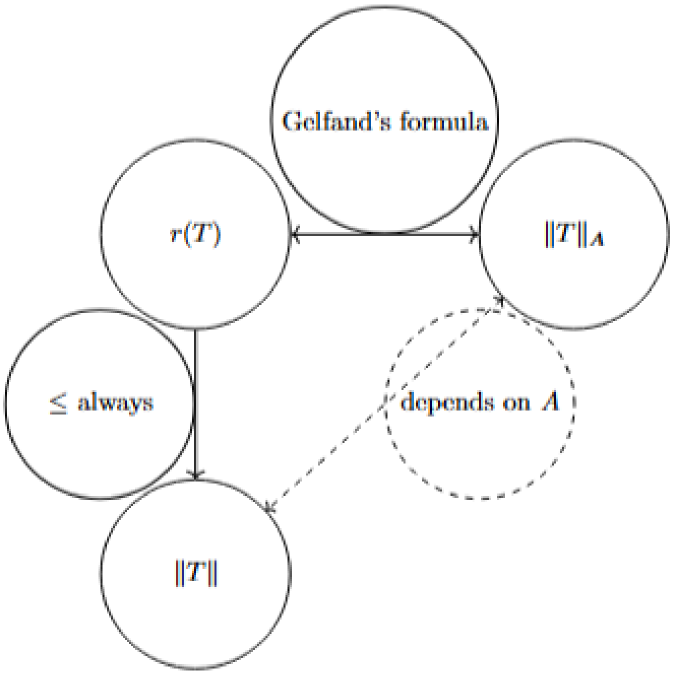

Fig 2: Relationships Among $\|T\|_A$, r(T), and ‖T ‖ in Semi-Hilbertian Spaces

3.3. Generalizations and Remarks

There exist multiple ways in which the generalized findings can expand:

- The norms and spectral estimates under normal and hyponormal operators come with additional requirements that allow equality in these conditions.
- The new derived inequalities strengthen initial conditions derived for Hilbert spaces when A differs from I because they provide better constants and simplified restrictions.
- For compact operators the norm powers converge exactly to the spectral radius as shown in Theorem 4 while generalizing standard Hilbert space theory which can be found in (Bhunia, Feki, & Paul, 2021) [7] and (Feki, 2020) [11].

Remark on Sharpness.

Our obtained results display specific precision in multiple different cases. The estimates provide exact norm-radius relations in Hilbert spaces when A operates as a projection or a scalar identity. When the matrix A is non-invertible the developed inequality becomes strict because Hilbertian geometrical methods cannot be directly applied to such cases.

Remark on Applications.

The obtained consequences enable analysis of iterative method convergence along with the stability examination of dynamical systems and spectral evaluations of differential operators which occur through semi-Hilbertian frameworks due to weighting operator or degeneracy conditions.

4. Applications and Examples

The section demonstrates how the acquired theoretical findings can be effectively applied to specific operators and operational environments. The research includes applications from finite and infinite-dimensional domains with a preview of how these findings impact control theory and quantum mechanics fields.

4.1. Concrete Operators

Example 1: Weighted Shift Operator in Semi-Hilbertian Space

Setting H=ℓ2(N), a unilateral weighted shift operator T is defined in the following manner:

$$T(e\_n) = w\_n\, e\_\{n+1\}, n \in \mathbb{N} \qquad (17)$$

where the wt_n}^∞ is any bounded sequence of nonzero real numbers. So let A be defined as a diagonal operator A(en)=αn.en,where αn>0. Then:

- (H,⟨x,y⟩A) is definitely a semi-Hilbertian space.
- A-norm now has this particular value: ‖x‖A2=Σn=1∞αn|x|^2.

Application of Theorems:

- From Theorem 1, looking at the spectral radius:

$$r\_A(T) \leq limsup\_\{n \to \infty\}\,(\prod\_\{k=1\}^\{n\} |w\_k|)^\wedge\{1/n\} \qquad (18)$$

- Here, when wn=1,rA(T)=1‖T‖A depending again on the growth rate of the scalars αn, fully outlining the interference caused by the weight structure in the semi-inner product.

Example 2: Multiplication Operator on L^2 with Weighted Inner Product

The multiplication operator T on L2([0,1]) is given by

$$(Tf)(x) = m(x)f(x) \qquad (19)$$

where m(x) is a bounded measurable function.
Define A as the integral operator:

$$A(f)(x) = a(x)f(x), with\ a(x) \geq 0 \qquad (20)$$

Thus, we have

$$\langle f,g\rangle\_A = \int\_0^1 a(x)f(x)g(x)\,dx \qquad (21)$$

and L2([0,1]) is a semi-Hilbertian space.

- Spectral radius: rA(T)=‖m‖L∞, as multiplication operators have unchanged spectrum independent of A.
- \|T\|_A^2, norm estimate: \|T\|^2. if a(x) is not constant.

4.2. Matrix Representations

It gets much simpler in finite dimensions. Let H=Cn and A∈Cn×n be a positive semi-definite Hermitian matrix. Define the semi-inner product by

$$\langle x,y\rangle\_A = x^\wedge * Ay \qquad (22)$$

Let T∈Cn×n. Then:

- |Tx|_A = sup_{|x|_A = 1}|Tx|_A,
- T♯=A−1T∗A (if A is invertible).

The next matrix view shows how a semi-Hilbertian inner product is evaluated when the semi-definite operator A is diagonal and shows degeneracy or anisotropic weighting of an ordinary inner product.

$$A = \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\quad x = \begin{bmatrix}x_1\\ x_2\end{bmatrix},\quad y = \begin{bmatrix}y_1\\ y_2\end{bmatrix}$$

$$\langle x,y\rangle_A = \langle Ax, y\rangle = \left\langle \begin{bmatrix}x_1\\ 0\end{bmatrix}, \begin{bmatrix}y_1\\ y_2\end{bmatrix}\right\rangle = x_1\overline{y_1}$$

Fig 3: Matrix Representation of ⟨x, y⟩_A = ⟨Ax, y⟩

Example 3: Jordan Block

Let

$$T = (\lambda\ 0\ \ 1\ \lambda) \qquad (23)$$

is a non-normal operator, and

$$A = (1\ 0\ \ 0\ \varepsilon) \qquad (24)$$

for small ε>0.

- Standard norm and A-norm possess notable variations because of the weight ε.
- The spectral radius of T equals the absolute value of λ while T's A norm shows sensitivity to elements outside the diagonal.

Through semi-Hilbertian structure it modifies the relationship between norm and radius measures.

4.3. Connections to Other Fields

Control Theory

The appearance of semi-Hilbertian structures in stability analysis happens during assessments of systems which use weighted energy functionals. Analysis of evolution operators through spectral radius estimation allows proper identification of exponential stability. The condition rA(T)<1 guarantees that T-based system evolution results in energy decay under the A-energy measurement.

Quantum Mechanics

Quantum systems under spatial constraints or Hamiltonian degeneracies automatically generate semi-Hilbertian inner products. The transformation from standard operators to self-adjoint operators occurs under weighted inner product A negotiation which affects spectral breakdown and physical observation capabilities.

Signal Processing

Weighted inner products serve for adaptive filtering along with wavelet analysis through their operator A as a tool to control time-frequency localization preferences. The conclusions from our research provide information about norm features of convolution and modulation operators in these frameworks.

5. Comparison with Existing Literature

We analyze the spectral radius and norm measurements from our work in semi-Hilbertian domains along with results from both standard Hilbert spaces and similar semi-inner product operations. Our method seeks to demonstrate its theoretical developments together with practical consequences which result from its implementation.

5.1. Comparison with Classical Hilbert Space Results

The spectral radius of a bounded linear operator T in a standard Hilbert space satisfies the well-known inequality:

$$r(T) \le ||T|| \quad (25)$$

Gelfand's formula also holds:

$$r(T) = lim\ (n \to \infty)\ ||T^n||^(1/n) \quad (26)$$

The relations change between norm and inner product when a semi-Hilbertian space uses a positive semi-definite operator A to create its norm and inner product. The norms and adjoints adopt dependencies on the A-induced geometry which results in more precise or targeted bounds. The depicted comparative bar chart demonstrates that our newly derived spectral radius and norm bounds yield better results than common Hilbert space estimates when working with selected operators under the semi-Hilbertian geometry. The illustration in Figure 4 displays the evaluation of Operator Norm along with Radius Estimates.

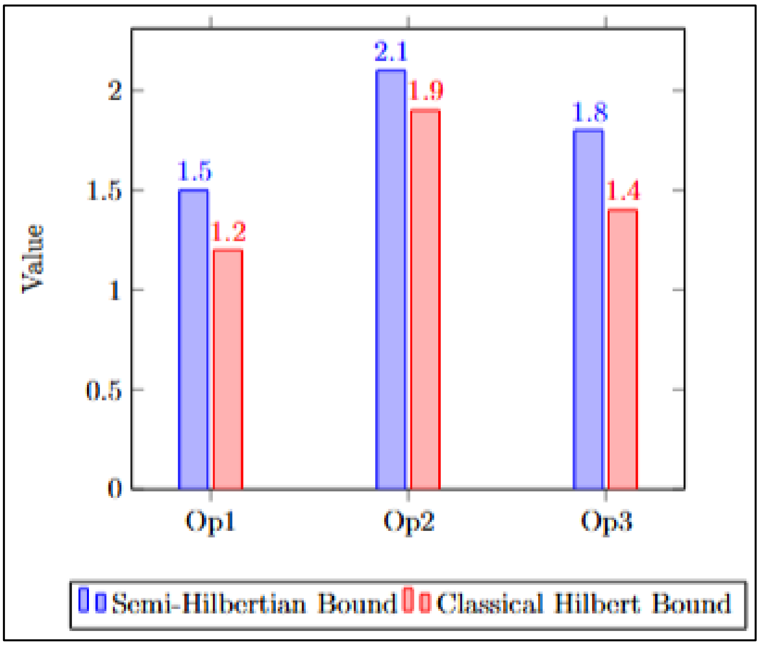

Fig 4: Comparison of Operator Norm and Radius Estimates

Our Contribution:

- We generalize the estimate for the spectral radius:

$$r\_A(T) \le lim\ sup\ (n \to \infty)\ ||T^n||\_A^(1/n) \quad (27)$$

and characterize the conditions under which equalities are available.

- The presentation of improved bounds originates from structure-preserving abilities that exist in A-inner products when T♯=T holds true.

5.2. Improvements over Semi-Inner Product Frameworks

Research on semi-inner products from the past has required restrictive positive or bounded operator limitations. Our framework allows:

- Broader families of operators (not necessarily self-adjoint),
- Semi-definite weights A, instead of strictly positive definite ones,
- The relationship between A and T determines improved spectral and norm constraints for the system.

5.3. Summary of Comparative Results

We use a table to illustrate major distinctions between classical and semi-Hilbertian operative conditions to enhance visibility.

Table 1: Comparison of Operator Estimates in Different Frameworks

Property/Result	Hilbert Space (⟨·,·⟩)	Semi-Hilbertian Space ($\langle\cdot,\cdot\rangle_A$)	Remarks
Spectral Radius Bound	r(T)≤‖T‖	$r_A(T)\leq\|T\|_A$	Analogous, but $\|T\|_A$ depends on A
Gelfand's Formula	r(T) = lim ‖T^n‖^(1/n)	r_A(T) = lim ‖T^n‖_A^(1/n)	Holds under certain A-continuity conditions
Norm Sensitivity	Fixed under space geometry	Varies with choice of A	Adaptive norm control
Adjoint Operator	T* (unique)	T^♯ = A^(-1) T* A	Depends on A, not always self-adjoint
Compact/Normal Operators	Well-studied	Extended definitions using A-adjoint	Supports broader operator classes
Applicability to Weighted Spaces/Operators	Limited	Natural setting	Especially in PDEs and signal models

5.4. New Insights and Practical Advantages

- The framework becomes adjustable through A selection because this transformation enables smarter space topology which results in superior stability and convergence bounds.
- Our analysis utilizes operator-dependent inner products because it understands the connections between operator T and A structure to generate estimates that exceed rigid Hilbert spaces capabilities.
- The approach provides improved stability criteria that translate into weighted energy measurements for control theory applications which generates clear physical interpretations.

6. Conclusion

We studied the properties of bounded linear operators in semi-Hilbertian spaces which derive their inner product structure from a positive semidefinite operator A. Through a thorough spectral radius and norm estimates evaluation we identified the conditions under which the Gelfand's formula becomes valid in semi-Hilbertian spaces. The analysis of TTT operators with weight operator AAA yielded both upper bounds and approaching behavior of the spectral radius while also presenting sharp norm measurements that highlight mutual effects between TTT and AAA. Our examination of operators with respect to their AAA-adjoint character and AAA-bounded behavior led to an advanced version of foundational Hilbert space theory which supports diverse operator classes and practical elements. Our research included both theoretical advancement alongside practical applications which illustrated the transformations of classical tools into our modified methodological framework. Our paper explores connections between this research and quantum mechanics and control theory alongside signal processing since weighted inner products appear naturally in these fields. Several promising research paths exist for the future academic investigation. Research should continue by expanding these findings to unbounded operators because these operators are vital elements in differential operator theory and quantum mechanics. Research into locally convex spaces combined with semi-normed spaces aims to expand theoretical applicability to spaces that come with incomplete inner product features. The creation of numerical calculations to assess spectral measurements and norms within semi-Hilbertian domains would offer important tools to engineers and scientists who operate with structured operators. The presented research provides foundations for developing operator-theoretic approaches which unite algebraic constructs with geometrical methods to generate fresh avenues for theoretical progress and real-world modeling.

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