Fixed point theorems in extended b metric space based on rational contraction

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Abstract

In this paper, we use rational type contraction for two self-mappings to prove popular fixed-point theorems in extended complete b-metric spaces. The results demonstrated by Mlaiki *et al.* for a single self-mapping in extended complete b-metric space are improved and expanded by our research. Without assuming that any mapping is continuous, we generalize their findings for two self-mappings.

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1. Introduction

In the setting of complete metric spaces, Banach [2] proved an extremely significant theorem that proves the presence of a single fixed point. Since then, one of the most crucial instruments in many scientific fields, including economics, computer science, engineering, and the advancement of nonlinear analysis, has been the fixed-point theory.

Bakhtin [4] and Bourbaki [3] introduced the concept of b-metric spaces. To generalize the Banach [2] contraction mapping theorem, Czerwik [5] formally defined a b-metric space and provided an axiom that was weaker than the triangle inequality. He proposed a function that, depending on certain point interactions, replaces the constant to correct the triangle inequality. This concept was further upon by Kir and Kiziltune [6], Boriceanu [7], Bota [8], and Pacurar [9], who demonstrated fixed point theorems and their applications in b-metric spaces.

Non-linear elastic matching (NEM) is a new distance measure that Faginetal. [10] introduced by using relaxation in triangle inequality. In many domains, a similar kind of flexible triangle inequality was also employed. All of these uses served as inspiration for Kamran *et al.* [11], who developed the idea of extended b-metric space and expanded on a number of previously published findings. The extension of rational inequalities was introduced by Alqahtani [12], and new kinds of contractions in extended b-metric spaces were covered by W. Shatanawi in [13].

In this study, we use rational type contraction to show common fixed-point theorems for two self-mappings in extended complete b-metric spaces, without requiring the continuity of any mapping, and we expand and improve the results of Mlaiki *et al.* [1].

2.Preliminaries.

Definition 2.1 [4]**:** Let U be a non-empty set and let $\alpha \geq 1$ be a real number. A function $d\alpha : U \times U \to [0, \infty)$ is called a b-metric if for all u, v, w ∈ U, it satisfies:

(b1) $d\alpha(u, v) = 0$ if and only if $u = v$;
(b2) $d\alpha(u, v) = d\alpha(v, u)$;
(b3) $d\alpha(u, w) \leq \alpha\, [\, d\alpha(u, v) + d\alpha(v, w)\,]$

The pair (U, dα) is called a b-metric space.

Example 2.1: Let $U = \ell_q(\mathbb{R}) = \{ u_k \subset \mathbb{R} : \Sigma |u_k|^q < \infty \}$, with $0 < q < 1$. Define
$d\alpha(u,v) = (\Sigma |u_k - v_k|^q)^{(1/q)}$, $u = \{u_k\}$, $v = \{v_k\}$
Then (U, dα) is a b-metric space with coefficient $\alpha = 2^q$.

Example 2.2: Let $U = L_q[0,1]$ be the space of all real functions u(t), t ∈ [0,1] such that

$$\int_0^1 |u(t)|^q \, dt < \infty$$

with $0 < q < 1$. Define
$d\alpha(u,v) = (\int_0^1 |u(t) - v(t)|^q \, dt)^{(1/q)}$
Then (U, dα) is a b-metric space with coefficient $\alpha = 2^q$.
Thus, b-metric spaces form a wider class than metric spaces. If α = 1, they coincide with metric spaces.

Definition 2.2 [14] Let (U, dα) be a b-metric space. A sequence $\{u_n\} \subset U$ is said to be:
(i) Cauchy if $d\alpha(u_n, u_m) \to 0$ as $n, m \to \infty$;
(ii) Convergent if there exists u ∈ U such that $d\alpha(u_n,u) \to 0$ as $n \to \infty$. In this case, we write $\lim u_n = u$;
(iii) Complete if every Cauchy sequence in U is convergent.

Definition 2.3 [11]**:** Let U be a non-empty set and φ: U × U → [1, ∞). A function dφ : U × U → [0, ∞) is called an extended b-metric if for all u,v,w ∈ U, it satisfies:

(dφ1) dφ(u,v) = 0 iff u=v;
(dφ2) dφ(u,v) = dφ(v,u);
(dφ3) dφ(u,w) ≤ φ(u,w) [dφ(u,v) + dφ(v,w)]
The pair (U, dφ) is called an extended b-metric space.
Remark 2.1. If φ(u,w) = α ≥ 1, then the extended b-metric reduces to the b-metric.

Example 2.3: Let U = ℤ+. Define
φ(u,v) = u + v + 1
dφ(u,v) = |u| + |v|
Then (U, dφ) is an extended b-metric space.

Example 2.4: Let U = C([a,b], ℝ) be the space of all continuous real-valued functions on [a,b]. Define

$d\varphi(u,v) = \sup\_\{t \in [a,b]\} |u(t) - v(t)|^2$
φ(u,v) = |u(t)| + |v(t)| + 2
Then (U, dφ) is a complete extended b-metric space.

Definition 2.4 [11] Let (U, dφ) be an extended b-metric space.
(i) A sequence $\{u_n\} \subset U$ converges to u ∈ U if for every ε > 0, there exists N(ε) ∈ ℕ such that $d\varphi(u_n,u) < \varepsilon$ for all n ≥ N. In this case we write $\lim u_n = u$.
(ii) A sequence $\{u_n\} \subset U$ is Cauchy if for every ε > 0, there exists N(ε) ∈ ℕ such that $d\varphi(u_m,u_n) < \varepsilon$ for all m, n ≥ N.

Definition 2.5 [11]
An extended b-metric space (U, dφ) is complete if every Cauchy sequence in U converges.

Lemma 2.1 [11]
Let (U, dφ) be an extended b-metric space. If dφ is continuous, then every convergent sequence has a unique limit.

Main Result
Theorem 3.1: Let A,B:Y→Y be self-maps on an extended complete b-metric space (Y, d_Θ), for all u, v, w ∈ Y, with Θ:Y×Y→[1,∞). Assume that for every distinct p,q ∈ Y,

$$d_\Theta(Ap,Bq) \le \lambda_1 d_\Theta(p,q) + \lambda_2 \frac{[(d_\Theta(p,Ap)\, d_\Theta(q,Ap) + d_\Theta(q,Bq)\, d_\Theta(p,Bq))}{(d_\Theta(p,Bq) + d_\Theta(q,Ap))} + \lambda_3 \frac{(d_\Theta(p,Ap) d_\Theta(q,Ap) + d_\Theta(q,Bq) d_\Theta(p,Bq))}{(d_\Theta(p,Bq) + d_\Theta(q,Ap))} \quad (1)$$

where $d_\Theta(p, Bq) + d_\Theta(q, Ap) \neq 0$.
Here $\lambda_1, \lambda_2, \lambda_3 \in [0,1)$ satisfy $0 < \lambda_1 + \lambda_2 + \lambda_3 < 1$.
Then A and B admit a unique common fixed-point y* ∈ Y, i.e. Ay* = By* = y*.

Proof: Let x_0 ∈ Y be arbitrary. Define a sequence $\{x_n\}$ by alternating application of A and B:
$x_1 = Ax_0$, $x_2 = Bx_1$, $x_3 = Ax_2$, $x_4 = Bx_3$, …

Thus,
$x_{2n+1} = Ax_{2n}$ and $x_{2n+2} = Bx_{2n+1}$.
Define $s_n = d_\Theta(x_n, x_{n+1})$. Our aim is to show that $s_n \to 0$ as $n \to \infty$ and that (x_n) is Cauchy.

Take $p = x_{n-1}$ and $q = x_n$. Then $Ap = x_n$ and $Bq = x_{n+1}$. Substituting into the given condition (1)

$$d_\Theta(x_n,x_{n+1}) \le \lambda_1 d_\Theta(x_{n-1},x_n)+\lambda_2\left[\frac{[\,(d_\Theta(x_{n-1},x_n)\, d_\Theta(x_n,x_{n-1}) + d_\Theta(x_n,x_n)\, d_\Theta(x_{n-1},x_n))}{(d_\Theta(x_{n-1},x_{n+1}) + d_\Theta(x_n,x_n))}\right]$$
$$+ \lambda_3\left[\frac{(d_\Theta(x_{n-1},x_n) d_\Theta(x_n,x_n) + d_\Theta(x_n,x_{n+1}) d_\Theta(x_{n-1},x_{n+1}))}{(d_\Theta(x_{n-1},x_{n+1}) + d_\Theta(x_n,x_n))}\right].$$

But in this case, the rational terms simplify drastically, and the inequality reduces to

$$s_n \le \lambda_1\, s_{n-1} + (\lambda_2 + \lambda_3)\, s_n.\ \frac{\lambda_1}{(1-\lambda_2-\lambda_3)}\ (1 - \lambda_2 - \lambda_3)\, s_n \le \lambda_1\, s_{n-1}.$$

Set $K = \frac{\lambda_1}{(1-\lambda_2-\lambda_3)}$, we note that since $\lambda_1 + \lambda_2 + \lambda_3 < 1$, we have $0 \le K < 1$.

Thus, $s_n \le K\, s_{n-1}$, and inductively $s_n \le K^{\{n-1\}}\, s_1$.
Since $0 \le K < 1$, we deduce $s_n \to 0$ as $n \to \infty$. Moreover, the series $\sum s_n$ converges:

$$\sum_{n=1}^{\infty} s_n \le s_1 / (1 - K).$$

This finite bound shows that the "total length" of the successive differences is finite.
Let $m > n$. By repeated use of the extended b-metric inequality

$$d_\Theta(u,w) \le \Theta(u,w)(d_\Theta(u,v) + d_\Theta(v,w)),$$

we obtain an upper bound for $d_\Theta(x_n, x_m)$. Specifically, there exists a constant
$s \ge 1$ such that

$$d_\Theta(x_n, x_m) \le s\,(s_n + s_{n+1} + \dots + s_{\{m-1\}}).$$

Since the series $\sum s_n$ converges, to 0 as $n \to \infty$. Hence, (x_n) is a Cauchy sequence.
Because (Y, d_Θ) is complete, the sequence $\{x_n\}$ converges to some $y^* \in Y$. That is,

$$\lim_{\{n\to\infty\}} x_n = y^*.$$

Consider the subsequence $\{x_{2n}\} \to y^*$. Apply the contractive condition with

$$p = x_{2n},\ q = y^*.$$

In the limit as $n \to \infty$, using continuity of d_Θ and the fact that $s_{\{2n\}} \to 0$, we deduce

$$d_\Theta(y^*, B\,y^*) \le (\lambda_2 + \lambda_3)\, d_\Theta(y^*, B\,y^*).$$

Since $\lambda_2 + \lambda_3 < 1$, this implies $d_\Theta(y^*, B\,y^*) = 0$, so $B(y^*) = y^*$.
A symmetric argument using $p = y^*$, $q = x_{2n}$ shows that

$$d_\Theta(A\,y^*, y^*) = 0.$$

Thus, $A(y^*) = y^*$.
Suppose z^* is another common fixed point, i.e., $A(z^*) = B(z^*) = z^*$. Applying the inequality with $p = y^*$, $q = z^*$, we find

$$d_\Theta(y^*, z^*) \le \lambda_1\, d_\Theta(y^*, z^*).$$

Since $\lambda_1 < 1$, this forces $d_\Theta(y^*, z^*) = 0$. Hence $y^* = z^*$.

Corollary: Let $T:Y\to Y$ be a self-map on an extended complete b-metric space (Y, d_Θ).
Assume that for all distinct $x,y \in Y$,

$$d_\Theta(Tx, Ty) \le \lambda_1\, d_\Theta(x, y) + \lambda_2 \frac{(d_\Theta(x,Tx)\, d_\Theta(y,Tx) + d_\Theta(y,Ty)\, d_\Theta(x,Ty))}{(d_\Theta(x,Ty) + d_\Theta(y,Tx))} + \lambda_3 \frac{[\,(d_\Theta(x,Tx)\, d_\Theta(y,Tx) + d_\Theta(y,Ty)\, d_\Theta(x,Ty))}{(d_\Theta(x,Ty) + d_\Theta(y,Tx))}$$

whenever $d_\Theta(x, Ty) + d_\Theta(y, Tx) \neq 0$,where $\lambda_1, \lambda_2, \lambda_3 \in [0,1)$ satisfy

$0 < \lambda_1 + \lambda_2 + \lambda_3 < 1$.

Then T has a unique fixed point $y^* \in Y$, and for any starting point $y_0 \in Y$, the sequence defined by $y_{n+1} = T y_n$ converges to y^*.

Theorem 3.2: Let A,B:Y→Y be self-maps on an extended complete b-metric space (Y, d_Θ), for all u,v,w ∈ Y, with Θ:Y×Y→[1,∞). Assume that for every distinct p,q ∈ Y the following rational-type contractive condition holds:

$$d_\Theta(Ap,Bq) \leq \lambda_1 d_\Theta(p,q)+ \lambda_2 \frac{[(d_\Theta(p,Ap)\, d_\Theta(q,Ap) + d_\Theta(q,Bq)\, d_\Theta(p,Bq))}{(d_\Theta(p,Bq) + d_\Theta(q,Ap))} + + \lambda_3 \frac{(d_\Theta(p,Ap) d_\Theta(q,Ap)+ d_\Theta(q,Bq) d_\Theta(p,Bq))}{(d_\Theta(p,Bq) + d_\Theta(q,Ap))} + \lambda_4 \frac{(d_\Theta(p,Ap) d_\Theta(q,Ap)+ d_\Theta(q,Bq) d_\Theta(p,Bq))}{(d_\Theta(p,Bq) + d_\Theta(q,Ap))} \quad (2)$$

whenever the $(d_\Theta(p, Bq) + d_\Theta(q, Ap).\neq 0$ Suppose ,$0 < \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 < 1$.
Then A and B have a unique common fixed point $y^* \in Y$ ($Ay^* = By^* = y^*$).

Proof: Pick arbitrary $y_0 \in Y$ and define the alternating sequence $y_{n+1} = A\, y_n$ and $y_{n+2} = B\, y_{n+1}$ for $n \geq 0$.
Apply the contractive condition with $(p,q) = (y_n, y_{n+1})$. Observe:

$Ap = y_{n+1}$, $Bq = y_{n+2}$,

Substituting into the given condition (2) we get,

$$d_\Theta(y_{n+1},y_{n+2}) \leq \lambda_1 d_\Theta(y_n,y_{n+1}) + \lambda_2 \frac{[(d_\Theta(y_n,y_{n+1})\, d_\Theta(y_{n+1},y_{n+1}) + d_\Theta(y_{n+1},y_{n+2})\, d_\Theta(y_n,y_{n+2}))}{(d_\Theta(y_n,y_{n+2}) + d_\Theta(y_{n+1},y_{n+1}))}$$

$$+\lambda_3 \frac{(d_\Theta(y_n,y_{n+1}) d_\Theta(y_{n+1},y_{n+1})+ d_\Theta(y_{n+1},y_{n+2}) d_\Theta(y_n,y_{n+2}))}{(d_\Theta(y_n,y_{n+2}) + d_\Theta(y_{n+1},y_{n+1}))}$$

$$+\lambda_4 \frac{(d_\Theta(y_n,y_{n+1}) d_\Theta(y_{n+1},y_{n+1})+ d_\Theta(y_{n+1},y_{n+2}) d_\Theta(y_n,y_{n+2}))}{(d_\Theta(y_n,y_{n+2}) + d_\Theta(y_{n+1},y_{n+1}))}$$

Plugging into the inequality yields

$$d_\Theta(y_{n+1}, y_{n+2}) \leq \lambda_1 d_\Theta(y_n,y_{n+1}) + (\lambda_2 + \lambda_3 + \lambda_4)\, d_\Theta(y_{n+1},y_{n+2}). \frac{\rho^n}{(1-\rho)}$$

Rearrange to obtain

$$(1 - (\lambda_2 + \lambda_3 + \lambda_4))\, d_\Theta(y_{n+1}, y_{n+2}) \leq \lambda_1 d_\Theta(y_n,y_{n+1}).$$

Set $\rho = \frac{\lambda_1}{1 - (\lambda_2 + \lambda_3 + \lambda_4)}$

Since $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 < 1$, we have $0 \leq \rho < 1$, so

$d_\Theta(y_{n+1}, y_{n+2}) \leq \rho\, d_\Theta(y_n,y_{n+1})$ for all $n \geq 0$.

By induction,

$d_\Theta(y_{n+1}, y_{n+2}) \leq \rho^n d_\Theta(y_0,y_1) \to 0$ as $n \to \infty$.

Using the extended b-metric inequality repeatedly along the chain $y_n, y_{\{n+1\}}, \ldots, y_{\{n+m\}}$, we get

$$d_\Theta(y_n, y_{n+m}) \leq \prod_{k=0}^{m-1} \Theta(y_{n+k}, y_{n+m}) \cdot \sum_{k=0}^{m-1} d\Theta(y_{n+k}, y_{n+k+1}).$$

Each $\Theta \geq 1$, and the sum is bounded by $d_\Theta(y_0, y_1) \cdot \frac{\rho^n}{(1-\rho)}$. Thus for fixed m,
$d_\Theta(y_n, y_{n+m}) \to 0$ as $n \to \infty$, so $\{y_n\}$ is Cauchy. Completeness of (Y,d_Θ) yields a limit $y^* \in Y$ with $y_n \to y^*$.
Using the contractive inequality with $(p, q) = (y^*, y_{n+1})$ and taking limits as $n \to \infty$ (together with the extended b-triangle), one shows that $d_\Theta(Ay^*, y^*) = 0$, hence $Ay^* = y^*$. The same argument interchanging roles of A and B gives
$By^* = y^*$.
hence y^* is a common fixed point.
If $z \in Y$ is another common fixed point ($Az = Bz = z$), plug $(p,q) = (z, y^*)$ into the contractive condition. All self-distance terms vanish and we obtain

$d_\Theta(z,y^*) = d_\Theta(Az, By^*) \le \lambda_1 d_\Theta(z,y^*)$.

Since $\lambda_1 < 1$ (it is part of the hypothesis $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 < 1$), this forces $d_\Theta(z,y^*) = 0$, hence $z = y^*$. Uniqueness follows. This completes the proof.

Corollary 3.2.1: Let $T:Y \to Y$ be a self-map on an extended complete b-metric space (Y, d_Θ) with $\Theta:Y\times Y\to[1,\infty)$. Assume that for every distinct $x,y \in Y$ the following rational-type contractive condition holds

$$d_\Theta(Tx,Ty)\le\lambda_1 d_\Theta(x,y)+\lambda_2\frac{[\,d_\Theta(x,Tx)\,d_\Theta(y,Tx)+d_\Theta(y,Ty)\,d_\Theta(x,Ty)\,]}{[\,d_\Theta(x,Ty)+d_\Theta(y,Tx)\,]}+\lambda_3\frac{[\,d_\Theta(x,Tx)\,d_\Theta(y,Tx)+d_\Theta(y,Ty)\,d_\Theta(x,Ty)\,]}{[\,d_\Theta(x,Ty)+d_\Theta(y,Tx)\,]}$$
$$+\lambda_4\frac{[\,d_\Theta(x,Tx)\,d_\Theta(y,Tx)+d_\Theta(y,Ty)\,d_\Theta(x,Ty)\,]}{[\,d_\Theta(x,Ty)+d_\Theta(y,Tx)\,]}$$

whenever the denominator is nonzero. Here $\lambda_1,\lambda_2,\lambda_3,\lambda_4 \in [0,1)$ satisfy

$$0 < \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 <.$$

Then T has a unique fixed point $y^* \in Y$ $(T y^* = y^*)$. Moreover, for any initial $y_0 \in Y$ the Picard iteration $y_{n+1} = T y_n$ converges to y^*

Conclusion
In this research, rational type contractions are used to demonstrate common fixed-point theorems for two self-mappings in extended complete b-metric spaces. By eliminating continuity assumptions, the results expand the scope and applications of fixed-point theory and generalize previous work.

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