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Thermodynamics of a Viscous Modified Chaplygin Gas Model in a Flat Universe

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Abstract
We examine the thermodynamic properties of a viscous fluid known as viscous modified Chaplygin gas within the context of the homogeneous isotropic universe model. Physical parameter behavior in viscous fluid is discussed in order to analyze the nature of the universe. Our analysis shows how changes in the bulk viscosity parameter affect the universe's thermodynamic characteristics. The specific heat formalism is used to analyze the validity of the third law of thermodynamics. Thermodynamic entities are also used to discuss the thermal equation of state. Thermodynamic stability is examined using isothermal, adiabatic, and specific heat conditions from classical thermodynamics. When appropriate parameters are selected, it is discovered that the fluid configuration under investigation expands adiabatically and is thermodynamically stable.

Keywords: Cosmology, Bulk Viscosity, Thermodynamics, Stability

1. Introduction

Numerous astronomical observations, including cosmic microwave background radiation (CMB) [1], high-redshift Type Ia supernovae (SNIa) [2, 6], and matter power spectra [7], strongly suggest that a dark sector, which is believed to be the source of the universe's accelerated expansion, accounts for nearly 96% of the universe's current total energy content. The nature of this acceleration has been extensively studied in relation to dark energy [1, 3, 8, 10]. This dark region is commonly thought to consist of two components: dark matter and dark energy. Numerous cosmological models have been developed to study the properties of this sector within the framework of Ein- stein's General Relativity. One of the strongest explanations for dark energy, which is produced by the universe's negative pressure, is Einstein's cosmological constant (Λ) [11, 15].

The cosmological principle, which implies that the world is homogeneous and isotropic, is thought to hold on sufficiently enormous cosmic scales. Assuming this, the universe's matter-energy content is often depicted as a perfect fluid, an idealized substance with uniform properties and no viscosity. One of the most frequently accepted theories for dark energy is the cosmological constant (Λ), which naturally arises from the framework of general relativity and quantum field theory. Other approaches, such dynamical dark energy models, argue that dark energy is dynamic and evolves over time [16–24]. These models include shortcomings and unresolved issues even if they provide a more thorough foundation. In response to these limitations, the Chaplygin gas has been proposed as a unique fluid model that might offer a coherent explanation of dark energy and dark matter.

We are considering the Chaplygin gas (CG) [25, 32] as a dark energy candidate. The original model has been expanded to the generalized Chaplygin gas (GCG) [33, 37] in order to remain consistent with observational evidence. Other modifications lead to models such as the modified Chaplygin gas (MCG) [38] and the modified cosmic Chaplygin gas (MCCG) [39]. Even though it is still consistent with data, a workable dark energy model should be able to approximate the cosmological constant model under certain circumstances [40]. Most models based on Chaplygin gas successfully address the problem of late-time cosmic acceleration. However, they often fail to solve the original singularity issue. As a result, very few models attempt to simultaneously address the late-time acceleration problems and the initial singularity. The generalized Chaplygin Gas (GCG) has the EoS, $p = -\frac{A}{\rho^{\alpha}}$.

The modified Chaplygin Gas (MCG) has the following equation of state:

$$p = B\rho - \frac{A}{\rho^{\alpha}}, \tag{1}$$

where ρ and p stand for the fluid's density and pressure, respectively, and A and B are the positive constants.
Moreover, bulk viscosity has been shown to be important for cosmology [41, 43]. An early indication of the importance of viscosity may have come from the Chaplygin gas, which was first hypothesized in [44] and later verified by further study in [45,50]. Refs. [43, 48] explicitly discuss the viscous modified cosmic Chaplygin gas model (VMCCG) and the viscous modified Chaplygin gas (VMCG) model, both of which account for time-dependent energy density. Additionally, in Ref. [46], the viscous Chaplygin gas model has been investigated in the context of a non-flat Friedmann-Robertson-Walker (FRW) universe.
The viscous modified Chaplygin gas (VMCG) theory has been extensively studied as a possible explanation for the observed accelerated expansion of the universe. This idea offers a coherent framework for understanding dark energy and dark matter. In this research, we study the combined effects of bulk viscosity and the Chaplygin gas in a flat Friedmann-Robertson-Walker (FRW) universe. As described in Ref. [51], the Chaplygin gas component further modifies the basic Friedmann equations 1 by adding bulk viscosity $\xi = \xi_0\rho^{\frac{1}{2}}$ is the model for the bulk viscous coefficient. The parameter $\alpha = {}^1$ is used in the equation of state in Ref. [52], and we utilize the same value in our investigation. We analyze the thermodynamic stability of the VMCG model under this assumption and find that the stability conditions are conditionally satisfied. We investigate whether the conditions $\left(\frac{\partial p}{\partial V}\right)_S < 0$, $\left(\frac{\partial p}{\partial V}\right)_T < 0$, and (b) $c_V > 0$ hold, using the methodology for the generalized in Santos *et.al.* [53], and modified Chaplygin gas models [54], based on the thermodynamic criteria described in Ref. [55]. Instantaneous thermodynamic stability requires these prerequisites. According to these references [53, 54, 56] and [57, 58], we examine a number of cosmological parameters in the VMCG model, such as the adiabatic speed of sound, pressure, equation of state, and deceleration parameter. Furthermore, the interaction between VMCG and $f\,(R,\,T)$ gravity [59] in the FRW framework is examined, where the modified Friedmann equations include time-dependent contributions to energy density and pressure from both dark energy and the Chaplygin gas.
Numerous studies have been conducted on cosmological models based on viscous modified Chaplygin gas (VMCG) and related frameworks. In Ref. [60], the VMCG model was examined in both classical and loop quantum cosmology (LQC), with a focus on its cosmic implications. Ref. [61] conducted an observational constraints analysis on the equation of state (EoS) parameters of the viscous generalized Chaplygin gas model using current background data, obtaining suitable values in agreement with observations. In [62], the viscous modified Cosmic Chaplygin gas (MCCG) model was analyzed in a flat Friedmann-Robertson-Walker (FRW) universe with a cosmological constant. The study looked at the effects of viscosity and the cosmological constant on the universe's evolution using the square of the speed of sound, and evaluated the stability of the model by comparing it to the Cardassian universe model. Ref. [63] focused on a VMCG model in both classical and LQC scenarios, with a detailed analysis of dynamical stability. The study provided promising predictions for the deceleration and EoS parameters as well as solutions to the coincidence problem and the time of cosmic acceleration. Additionally, a modified Chaplygin gas model with bulk viscosity in a (2+1)-dimensional FRW space-time was studied in Ref. [64]. Several variations of the bulk viscosity coefficient were investigated in order to establish key physical parameters such as energy density, the Hubble parameter, and the deceleration parameter. The stability of the model was assessed again using the speed of sound. Finally, in Ref. [65], a generalized Chaplygin gas was explained in terms of bulk viscosity without the need for a cosmological constant. The study discovered that dark matter and dark energy are naturally connected when bulk viscosity is applied to standard FRW cosmology.

2. Field Equations of FRW Cosmology

We know that the Friedmann-Robertson-Walker measure has the following expression:

$$ds^2 = -dt^2 + a^2(t)\left(\frac{dr^2}{1-kr^2} + r^2 d\Omega^2\right) \tag{2}$$

where $d\Omega^2 = d\theta^2 + sin^2\theta d\varphi^2$, and $a(t)$ is the universe's expansion rate. Comoving coordinates include the dimensionless coordinates r, θ, and φ, as well as k = 0, 1, and -1, which represent flat, closed, and open universes, respectively. In this case, since we are studying in flat spacetime, ($k = 0$), the Einstein equation can be written as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}, \tag{3}$$

where the parameters are $\Lambda = 0$, $8\pi G = 1$, and $c = 1$. We found that the viscous modified Chaplygin gas (VMCG) of the universe acts as dark energy with EoS (1). Equations (2) and (3) [47, 49] yield the following result for bulk viscous fluid:

$$T_{\mu\nu} = (\rho + p_{vmcg})u_\mu u_\nu - p_{vmcg}g_{\mu\nu} \tag{4}$$

where the four-velocity vector is represented by u^μ and the energy-density vector by ρ. The universe's energy-momentum tensor is $T_{\mu\nu}$, the metric tensor is $g_{\mu\nu}$, and the normalization condition is $u^\mu u_\nu = 1$. Thus, in the VMCG model, the total effective pressure of the viscous fluid can be expressed as

$$p_{vmcg} = p + \Pi = B\rho - \frac{A}{\rho^{\alpha}} - \sqrt{3}\xi_0\rho \quad (5)$$

where ξ (= $\xi_0\rho$ 2) is the bulk viscosity coefficient and Π (= $3\xi H$) is the viscous pressure.
As stated in the definition, the energy density is

$$\rho_{vmcg} = \frac{U}{V}, \quad (6)$$

where V stands for the fluid's volume and U for its internal energy. The thermodynamics relation gives us

$$\left(\frac{\partial U}{\partial V}\right)_S = -p_{vmcg} \quad (7)$$

We obtain the relation from equations (1), (5), (6), and (7)

$$\left(\frac{\partial U}{\partial V}\right)_S = A\left(\frac{V}{U}\right)^{\alpha} - (B - \sqrt{3}\xi_0)\frac{U}{V}. \quad (8)$$

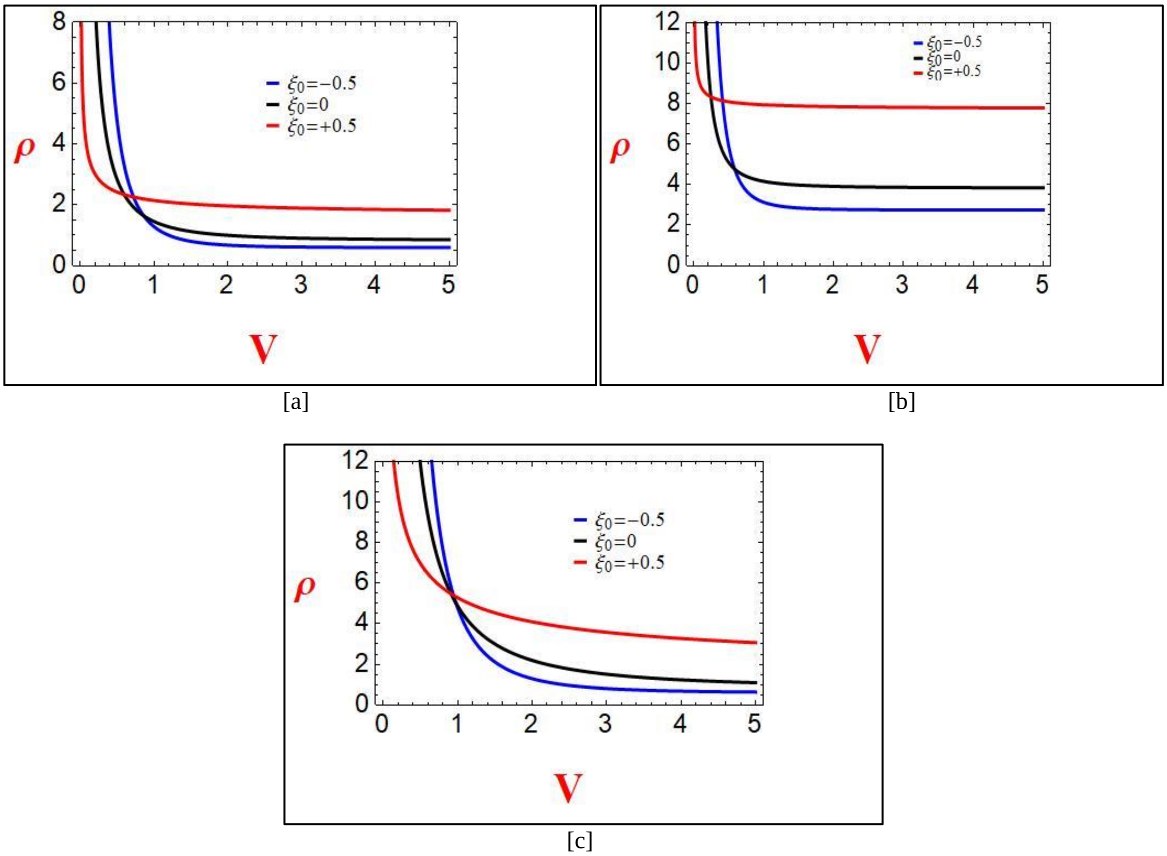


Fig 1: The plot of dark energy density ρ as a function of volume V in different values of viscous coefficient ξ_0, where we have taken for (a) $A = 1$, $B = 1/3$, $\alpha = 0.5$ & $b = 1$; (b) $A = 10$, $B = 1/3$, $\alpha = 0.5$ & $b = 1$; (c) $A = 1$, $B = 1/3$, $\alpha = 0.5$ & $b = 10$.

After integrating the previously described equation, the expression for internal energy U is

$$U = \left[\frac{A}{1 + B - \sqrt{3}\xi_0} V^{\alpha+1} + \frac{b}{V^{(B-\sqrt{3}\xi_0)(\alpha+1)}}\right]^{\frac{1}{\alpha+1}} \quad (9)$$

where an arbitrary integration constant is denoted by b. The expression given above can alternatively be expressed as

$$U=\left(\frac{A}{1+B-\sqrt{3}\xi_0}\right)^{\frac{1}{\alpha+1}}V\left[1+\left(\frac{\epsilon}{V}\right)^{(1+B-\sqrt{3}\xi_0)(\alpha+1)}\right]^{\frac{1}{\alpha+1}} \tag{10}$$

where,

$$\epsilon=\left[\frac{b(1+B-\sqrt{3}\xi_0)}{A}\right]^{\frac{1}{(1+B-\sqrt{3}\xi_0)(\alpha+1)}} \tag{11}$$

Consequently, the parameter ξ_0 can also be used to represent the energy-density as

$$\rho_{vmcg}=\left(\frac{A}{1+B-\sqrt{3}\xi_0}\right)^{\frac{1}{\alpha+1}}\left[1+\left(\frac{\epsilon}{V}\right)^{(1+B-\sqrt{3}\xi_0)(\alpha+1)}\right]^{\frac{1}{\alpha+1}} \tag{12}$$

Now, it is evident from equation (12) that the energy density becomes extremely high (i.e., $\rho_{vmcg}\to\infty$) at very low volumes, i.e., for $V\to 0$. On the other hand, the
viscous fluid's energy density is extremely low (i.e., ρ_{vmcg} *limit of very high volume, i.e.,* $V\to\infty$ *as shown in Fig.1.*

2.1. Pressure

We compute the effective pressure of the VMCG model with the volume V and the viscous parameter ξ_0. Equations (5) and (12) are utilized to determine

$$p_{vmcg}=\rho_{vmcg}\left[(B-\sqrt{3}\xi_0)-\frac{(1+B-\sqrt{3}\xi_0)}{\left[1+\left(\frac{\epsilon}{V}\right)^{(1+B-\sqrt{3}\xi_0)(\alpha+1)}\right]}\right] \tag{13}$$

It can also be expressed as

$$p_{vmcg}=-\left(\frac{A}{1+B-\sqrt{3}\xi_0}\right)^{\frac{1}{\alpha+1}}\frac{\left[1-(B-\sqrt{3}\xi_0)\left(\frac{\epsilon}{V}\right)^{(1+B-\sqrt{3}\xi_0)(\alpha+1)}\right]}{\left[1+\left(\frac{\epsilon}{V}\right)^{(1+B-\sqrt{3}\xi_0)(\alpha+1)}\right]^{\frac{\alpha}{\alpha+1}}} \tag{14}$$

The effective pressure of the VMCG model is represented by the previously described equation. The overall effective pressure turns negative when the bulk viscous pressure phrase becomes more prominent and negative than the thermodynamic pressure. Dark energy is characterized as the 'negative pressure' or tension that accelerates the universe's seeming rapid expansion. In context with the scenario, we currently have relations:

(a) When $\xi_0 = 0$, $B = 0$, $A\neq0$, and $\alpha\neq 0$, the equation (14) becomes,

$$p_{vmcg}=-\frac{(A)^{\frac{1}{\alpha+1}}}{\left[1+\left(\frac{\epsilon}{V}\right)^{(\alpha+1)}\right]^{\frac{\alpha}{\alpha+1}}}. \tag{15}$$

As expected, this is similar to the generalized Chaplygin gas model [53]. The effective pressure and the thermodynamic pressure are identical. In an expanding universe, density and pressure usually decrease as volume increases.

(b) When $\xi_0 = 0$, $A\neq 0$, $B\neq 0$, and $\alpha\neq 0$, the expression (14) becomes,

$$p_{vmcg}=-\left(\frac{A}{1+B}\right)^{\frac{1}{\alpha+1}}\frac{\left[1-B\left(\frac{\epsilon}{V}\right)^{(1+B)(\alpha+1)}\right]}{\left[1+\left(\frac{\epsilon}{V}\right)^{(1+B)(\alpha+1)}\right]^{\frac{\alpha}{\alpha+1}}}. \tag{16}$$

As predicted, this is similar to the modified Chaplygin gas model [54]. The effective pressure and the thermodynamic pressure are equivalent. In an expanding universe, density and pressure usually decrease with increasing volume.

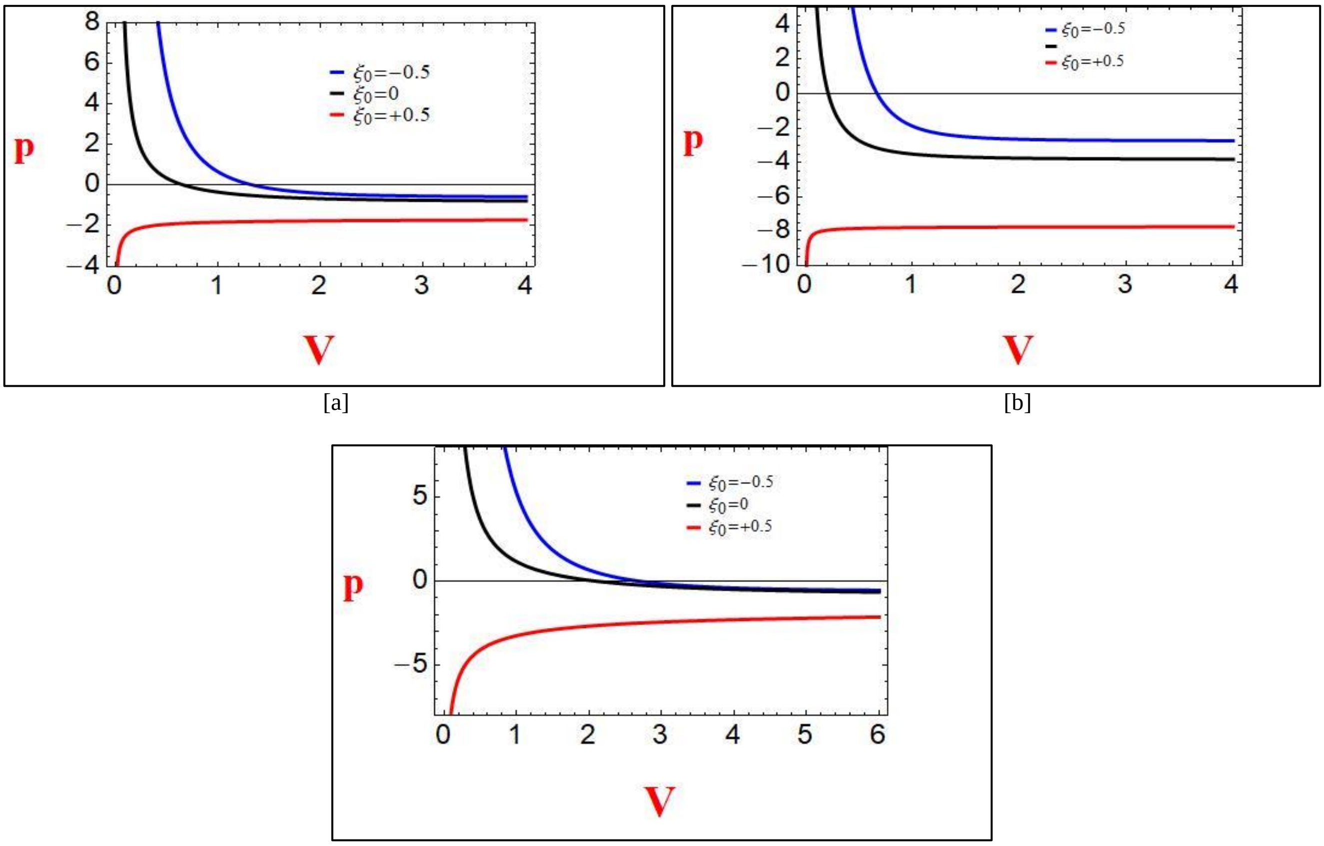


Fig 2: The plot of pressure p as a function of volume V in different values of viscous coefficient ξ_0, where we have taken for (a) $A = 1$, $B = 1/3$, $\alpha = 0.5$ & $b = 1$; (b) $A = 10$, $B = 1/3$, $\alpha = 0.5$ & $b = 1$; (c) $A = 1$, $B = 1/3$, $\alpha = 0.5$ & $b = 10$; (c) For $\xi_0 = 0$, $\alpha = -\frac{1}{2}$, $A \neq 0$ and $B \neq 0$, the model illustrates the emergent universe model [61], and (14) evolves becomes

$$p_{vmcg} = -\left(\frac{A}{1+B}\right)^{\frac{1}{2}} \frac{\left[1 - B\left(\frac{\epsilon}{V}\right)^{\frac{1+B}{2}}\right]}{\left[1+\left(\frac{\epsilon}{V}\right)^{\frac{1+B}{2}}\right]^{-1}} \tag{17}$$

For various values of the viscous parameter ξ_0, we plot pressure as a function of volume V, as shown in Fig.2. The thermodynamic pressure for" dust" (non-relativistic) is essentially insignificant. Consequently, a zero effective pressure is linked to decelerated expansion. The value of the parameter determines whether the effective pressure is positive or negative. For $\xi_0 = 0$ and $\xi_0 > 0$, pressure is always negative, suggesting that the gas is of Chaplygin type. It also shows that the effective pressure is positive for small volumes and negative for big volumes when $\xi_0 < 0$. The universe may become negative as its volume increases, acting as a source of repulsive gravity that speeds up expansion.

For the zero-pressure condition of the VMCG model, or $p_{vmcg} = 0$, the critical volume (V_c) is found as

$$V_c = \epsilon(B - \sqrt{3}\xi_0)^{\frac{1}{(1+B-\sqrt{3}\xi_0)(\alpha+1)}} \tag{18}$$

Alternatively,

$$V_c = \left[\frac{b(1+B-\sqrt{3}\xi_0)(B-\sqrt{3}\xi_0)}{A}\right]^{\frac{1}{(1+B-\sqrt{3}\xi_0)(\alpha+1)}} \tag{19}$$

When the critical volume is larger than volume V, i.e., $V_c > V$, an acceptable decelerated universe is indicated by a positive effective pressure (p_{vmcg}); for $V = V_c$, $p_{vmcg} = 0$. When $V > V_c$, effective pressure becomes negative, indicating exotic matter dominance and the universe is accelerating. When a dust-dominated universe reaches the acceleration phase, we derive a new scale of V_c from the study. We find that V_c and ϱ are of the same order of magnitude. $V \ll \epsilon$ represents the decelerated universe, while $V \gg \epsilon$ represents the extremely large volume with a potentially accelerated universe.

2.2. EoS Parameter

This section determines and defines the EoS of the bulk viscous fluid as

$$\omega_{vmcg} = \frac{p_{vmcg}}{\rho_{vmcg}} \tag{20}$$

The expression described above can also be expressed using equations (12) and (14):

$$\omega_{vmcg} = (B - \sqrt{3}\xi_0) - \frac{1 + B - \sqrt{3}\xi_0}{\left[1 + (\frac{\epsilon}{V})^{(1+B-\sqrt{3}\xi_0)(\alpha+1)}\right]} \tag{21}$$

The dynamic link between the effective EoS parameter and volume in a viscous cosmology is shaped by the bulk viscosity of the cosmic fluid. As the cosmic scale factor (a) varies, the effective EoS parameter does not remain constant. When the universe is thick in the early periods, bulk viscosity has little effect on the EoS. As the universe becomes denser and expands, the bulk viscosity may become dynamically important, driving the effective EoS toward negative values ($\omega_{vmcg} < 1/3$), which speeds up expansion.

(i) When volume is very small ($V \ll \epsilon$), i.e., $\frac{\epsilon}{V} > 1$, the EoS, $\omega_{vmcg} = B - \sqrt{3}\xi_0$, *and also the effective pressure*, $p_{vmcg} = (B - \sqrt{3}\xi_0)\rho_{vmcg}$. It is observed that when B ≤ 1 and $\xi_0 \geq \frac{3}{2}$, $\omega_{vmcg} < -1$, represents the phantom-like universe, and a quiescence phenomenon is indicated if B ≥ 1 and $\xi_0 \leq 1$, we get $\omega_{vmcg} > -1$. In this instance as well, if $\xi_0 = 0$, we obtain ω_{vmcg} = B, which reveals that p_{vmcg} = Bρ_{vmcg}, so this indicates that, in this region, the fluid behaves like a barotropic fluid.

(ii) When volume is very large, ($V \gg \epsilon$), i.e., $\frac{\epsilon}{V} = 0$, the EoS parameter is also obtained as, $\omega_{vmcg} \approx -1$. This indicates that the ΛCDM model. As seen in Fig.3, we plot effective EoS versus volume for a range of ξ_0 values, indicating that the universe is accelerating at a huge volume.

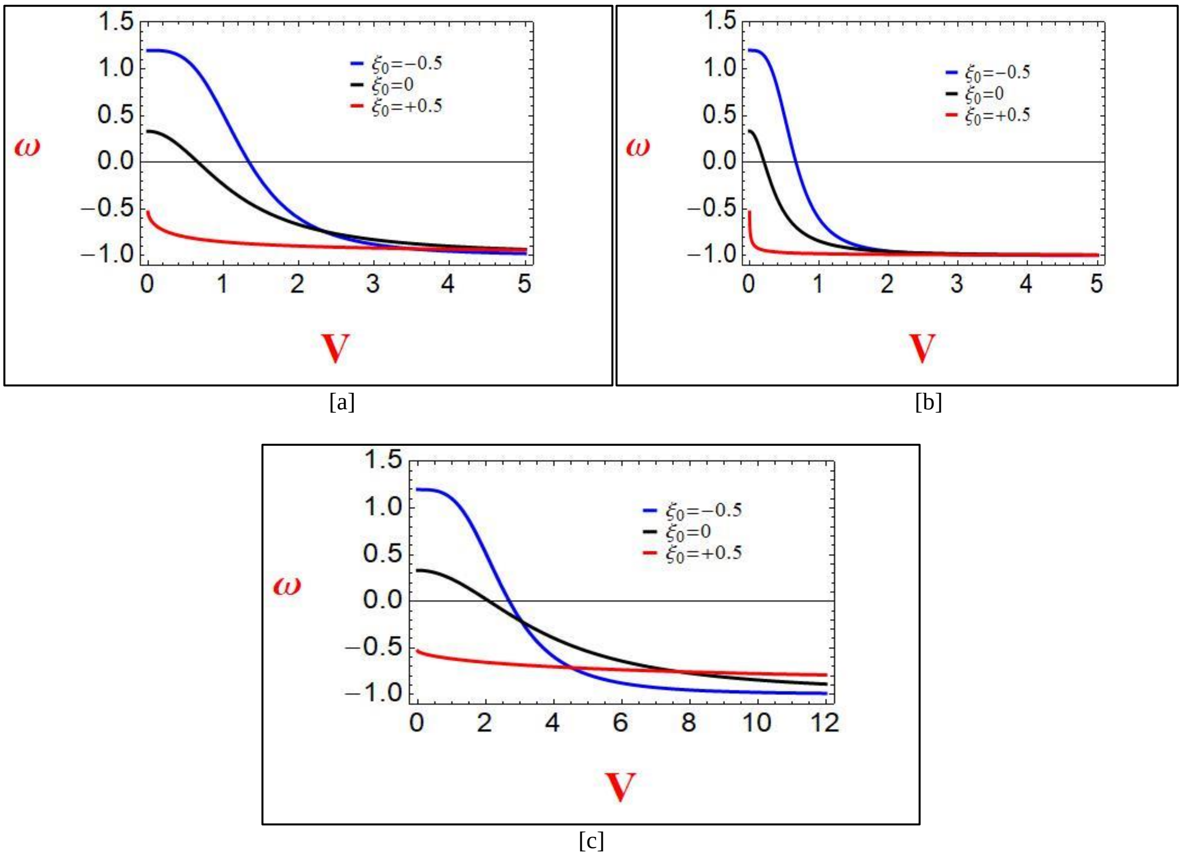


Fig 3: The plot of EoS parameter ω as a function of volume V in different values of viscous coefficient ξ_0, where we have taken for (a) A = 1, B = 1/3, α = 0.5 & b = 1; (b) A = 10, B = 1/3, α = 0.5 & b = 1; (c) A = 1, B = 1/3, α = 0.5 & b = 10.

In the early universe, when the volume is comparatively very small and ωvmcg tends to zero when V rises to Vc, $\omega_{vmcg} > 0$ for $\xi_0 < 0$. So, we also find $V = V_c = \left[\frac{b(-\sqrt{3}\xi_0)(1-\sqrt{3}\xi_0)}{A}\right]^{\frac{1}{(1-\sqrt{3}\xi_0)(\alpha+1)}} = \epsilon\left[-\sqrt{3}\xi_0\right]^{\frac{1}{(1-\sqrt{3}\xi_0)(\alpha+1)}}$, $\omega_{vmcg} = 0$. Once more, V increases and ω_{vmcg} becomes negative. From Fig.3c, we can observe that $\xi_0 = 0$, $\omega_{vmcg} = -1$, i.e., the popular ΛCDM model. However, for big volume, we obtain $0 > \omega_{vmcg} > -1$ for $\xi_0 > 0$. A similar outcome was previously demonstrated in a higher dimensional example as well [66].

2.3. Deceleration Parameter

Based on the specification, the effective deceleration parameter that corresponds to the bulk viscosity of the VMCG model can be expressed as

$$q_{vmcg} = \frac{1}{2} + \frac{3}{2}\frac{p_{vmcg}}{\rho_{vmcg}} \tag{22}$$

The equation can be written using the viscous parameter ξ_0 using equation (20) as

$$q_{vmcg} = \frac{1}{2} + \frac{3}{2}\left[(B-\sqrt{3}\xi_0) - \frac{1+B-\sqrt{3}\xi_0}{\left[1+(\frac{\epsilon}{V})^{(1+B-\sqrt{3}\xi_0)(\alpha+1)}\right]}\right] \tag{23}$$

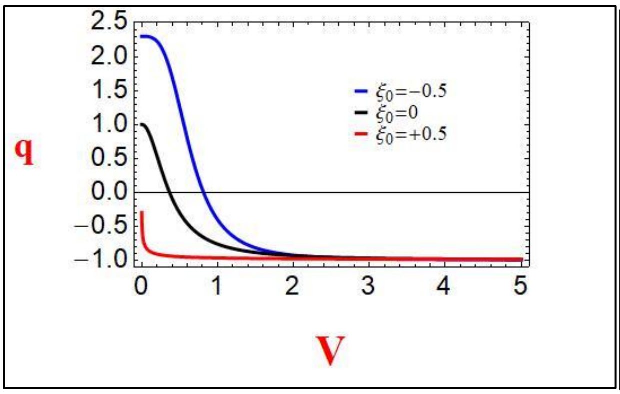
The deceleration parameter in a cosmology where a viscous fluid predominates dynamically changes with the universe's volume, which is represented by the scale factor. As a result of cosmic acceleration driven by viscosity, the deceleration parameter changes from a positive (decelerating) value in the early universe to a negative (accelerating) value in the later universe. In order to explain the observable expansion history, this dynamic behavior offers an alternative to the cosmological constant (Λ). Radiation and matter densities were quite high in the early cosmos. In a universe dominated by matter, where viscosity is zero, ($q = 0.5$). Because of the large energy density and initial insignificant negative pressure associated with bulk viscosity, q is positive, signifying a decelerating phase. Energy density falls as the universe gets bigger and its volume rises. However, the negative viscous pressure may start to dominate. As the cosmic volume expands, the negative bulk viscous pressure becomes more significant. In order to move from a decelerating to an accelerating phase, the deceleration parameter falls, exceeds the threshold ($q = 0$) (which corresponds to a constant expansion rate), and becomes negative ($q < 0$).

(i) For small volume (early universe), $V \ll \epsilon$, i.e., $\frac{\epsilon}{V} = \infty$, we get $q_{vmcg} = \frac{1}{2} - \frac{3\sqrt{3}\xi_0}{2}$. The effective deceleration parameter q_{vmcg} is positive for $\xi_0 \leq 0$ when volume is minimal, suggesting that the universe may be decelerating.

(ii) When large volume (late universe), $V \gg \epsilon$, i.e., $\frac{\epsilon}{V} = 0$, the universe accelerates. This is indicated by the value $q_{vmcg} \approx -1$. In this instance, the *flip* volume (V_f) will occur when the effective deceleration parameter is zero. In many viscous models, the deceleration parameter gets closer to $q = 1$ as the scale factor gets closer to infinity. This is equivalent to a de Sitter universe, which behaves like a cosmological constant and experiences exponential growth driven by a constant vacuum energy density. Consequently, the flip volume expression can be stated as

$$V_f = \left[\frac{b(1+B-\sqrt{3}\xi_0)(1+3B-3\sqrt{3}\xi_0)}{2A}\right]^{\frac{1}{(1+B-\sqrt{3}\xi_0)(\alpha+1)}} \tag{24}$$

According to Fig.4, the cosmos accelerates as volumes rise after q_{vmcg} initially goes to zero. The equation above demonstrates that the value of V_f must be genuine when $\xi_0 < 0$; otherwise, there won't be any *flip*. Accordingly, the universe accelerates when $V > V_f$ and decelerates when $V < V_f$. Therefore, we determine that two scales of volume are V_c and V_f, respectively, from the zero-pressure condition, and that a potential deceleration-acceleration transition takes place at the effective deceleration parameter.



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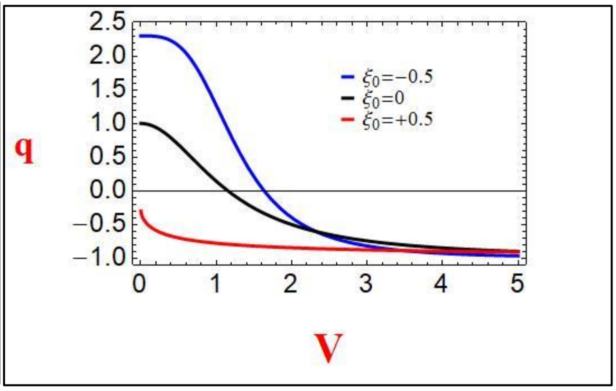

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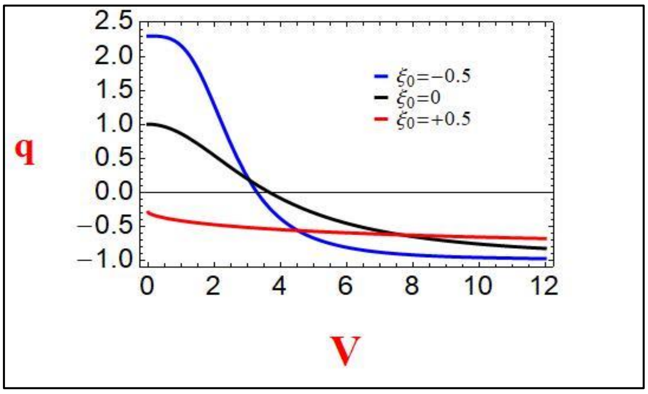


[c]

Fig 4: The plot of deceleration parameter q as a function of volume V in different values of viscous coefficient ξ_0, where we have taken for (a) $A = 1$, $B = 1/3$, $\alpha = 0.5$ & $b = 1$; (b) $A = 10$, $B = 1/3$, $\alpha = 0.5$ & $b = 1$; (c) $A = 1$, $B = 1/3$, $\alpha = 0.5$ & $b = 10$.

We understand that for a flip to occur in a FRW cosmology, pressure must be both negative and less than $\frac{\rho}{3}$ (*i.e.,* $\rho + 3p < 0$). Equations (19) and (24) make it evident that in this instance, $V_c < V_f$. Additionally, it can be demonstrated that the density-pressure requirement $\rho + 3p = 0$ is satisfied for this specific volume $V = V_f$.
We now obtain the relation from equations (18) and (23) as

$$\frac{V_f}{V_c} = \left[\frac{1+3B-3\sqrt{3}\xi_0}{2(B-\sqrt{3}\xi_0)}\right]^{\frac{1}{(1+B-\sqrt{3}\xi_0)(\alpha+1)}} \tag{25}$$

It is evident from the above expression that $V_f > V_c$ holds for $0 < B < 1$ and $\xi_0 < 0$.

2.4 Speed of Sound

Here, we looked at the VMCG model's stability condition. The definition of sound speed in a viscous fluid is

$$v_s^2 = \left(\frac{dp_{vmcg}}{d\rho_{vmcg}}\right)_S = B - \sqrt{3}\xi_0 + \frac{\alpha(1+B-\sqrt{3}\xi_0)}{\left[1+(\frac{\epsilon}{V})^{(1+B-\sqrt{3}\xi_0)(\alpha+1)}\right]} \tag{26}$$

The velocity of sound in the cosmic fluid is a dynamically changing quantity that is governed by the bulk viscosity and thermodynamic parameters of the fluid rather than being a constant in viscous cosmology. The relation between sound speed and volume is driven by the expansion of the universe, as opposed to static fluids where density and elastic characteristics control sound speed. Sound speed is comparatively stable in the early cosmos because adiabatic processes dominate the fluid's characteristics. In the late universe, bulk viscosity increases with increasing volume, causing the sound speed to dynamically vary and possibly turn negative. This is a prerequisite for the fluid to function as a source of rapid expansion.
We also know that $0 < v_s^2 < 1$ must be the range of the speed of sound. This range was the focus of our investigation.
(i) When volume is very small ($V \ll \epsilon$), i.e., at early universe, the equation (26) gives $v_s^2 = B - \sqrt{3}\xi_0$, and for $\xi_0 = 0$ which leads to $0 < B < 1$, and this limit includes $B = \frac{1}{3}$, the radiation dominated universe.
(ii) When volume is very large ($V \gg \epsilon$), equation (26) yields

$$v_s^2 = \alpha + (B - \sqrt{3}\xi_0)(\alpha + 1) \tag{27}$$

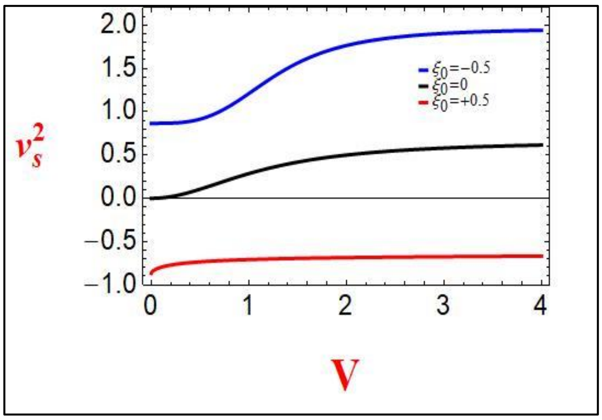
In the equation above, ξ_0 and α are necessary. The phantom type universe is the result of the thermodynamic stability condition for values of $\alpha = \frac{1}{2}$ and $\xi_0 > \frac{1}{\sqrt{3}}$, as we shall demonstrate later [67, 68]. Furthermore, for $\alpha > 0$, the foregoing calculation yields an imaginary speed of sound, resulting in a perturbative cosmology. We discovered in ref. [69] that for holographic DE, the sound speed is always negative and non-negative for tachyon matter and generic Chaplygin gas. As seen in Fig.5, the properties of the squared speed of sound change depending on the value of the parameter ξ_0 *for* α = ½. It is evident that α = ½ yields thermodynamically stable results throughout the range1. Panigrahi & Chatterjee [58] also looked at the variable MCG model, which allows for both positive and negative sound speed values. With bulk viscosity playing a major role in its evolution, the effective sound speed is an essential measure for assessing the stability and behavior of cosmic disturbances.

3. Thermodynamical Stability

We examine a fluid's thermodynamic stability conditions throughout the universe's evolution. The stability requirements of thermodynamics are known to be [55]: In both adiabatic and isothermal expansion, the pressure decreases as $\left(\frac{\partial p}{\partial V}\right)_S < 0$, $\left(\frac{\partial p}{\partial V}\right)_T < 0$, and (b) $c_V > 0$.

Differentiating equation (14) w.r.t. volume (V), we obtained as

$$\left(\frac{\partial p_{vmcg}}{\partial V}\right)_S = \frac{p_{vmcg}}{V}\frac{(1+B-\sqrt{3}\xi_0)}{\left[1+\left(\frac{\epsilon}{V}\right)^{-(1+B-\sqrt{3}\xi_0)(\alpha+1)}\right]} \times \frac{\left[\alpha+(B-\sqrt{3}\xi_0)\left[1+\alpha+\left(\frac{\epsilon}{V}\right)^{(1+B-\sqrt{3}\xi_0)(\alpha+1)}\right]\right]}{\left[1-(B-\sqrt{3}\xi_0)\left(\frac{\epsilon}{V}\right)^{(1+B-\sqrt{3}\xi_0)(\alpha+1)}\right]} \tag{28}$$



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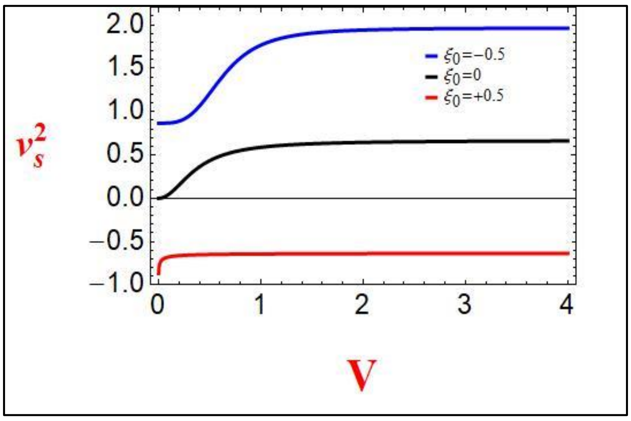

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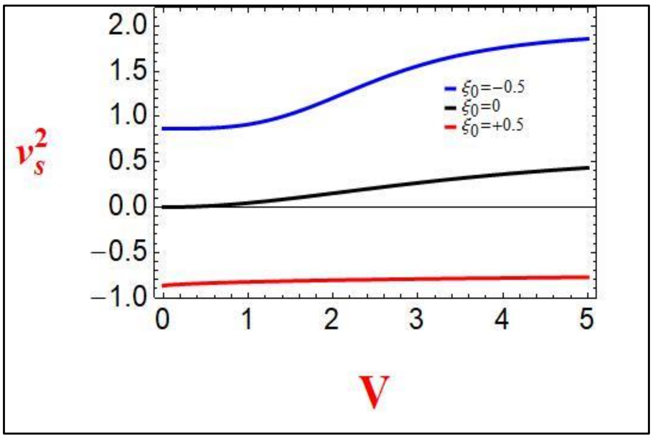

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Fig 5: The plot of square speed of sound v2 as a function of volume V in different values of viscous coefficient ξ0, where we have taken for (a) A = 1, B = 1/3, α = 0.5 & b = 1; (b) A = 10, B = 1/3, α = 0.5 & b = 1; (c) A = 1, B = 1/3, α = 0.5 & b = 10.

(1) When volume is very small ($V \ll \epsilon$), the above expression can be written as $\left(\frac{\partial p_{vmcg}}{\partial V}\right)_S \approx -\frac{(1+B-\sqrt{3}\xi_0)p_{vmcg}}{V}$. According to earlier research, in the early universe, $p_{vmcg} = (B-\sqrt{3}\xi_0)\rho_{vmcg}$, therefore, in this case $\left(\frac{\partial p_{vmcg}}{\partial V}\right)_S \approx -\frac{(1+B-\sqrt{3}\xi_0)(B-\sqrt{3}\xi_0)\rho_{vmcg}}{V}$ which is independent of A but very much depends on ξ0. Hence, the pressure in this evolution is negative, therefore we can get $\left(\frac{\partial p}{\partial V}\right)_S < 0$ for α > 0.

(2) For large volume ($V \gg \epsilon$), the above equation (28) reduces to zero, i.e., ∂pvmcg ≈ 0 which indicate that the pressure is constant at late universe.

To limit the parameters used here, we now go over a few special instances:

(i) For ξ0 = 0, B = 0, α ≠ 0, we get the equation (28) becomes

$$\left(\frac{\partial p_{vmcg}}{\partial V}\right)_S = \alpha\frac{p_{vmcg}}{V}\left(\frac{\epsilon}{V}\right)^{\alpha+1}\left[1+\left(\frac{\epsilon}{V}\right)^{\alpha+1}\right]^{-1} \tag{29}$$

which the above expression is similar like to the generalized Chaplygin gas (GCG) model of Santos [53].

(ii) For $\xi_0 = 0$, $A \neq 0$, $B \neq 0$, and $\alpha \neq 0$, the equation (28) reduces to

$$\left(\frac{\partial p_{vmcg}}{\partial V}\right)_S = \frac{p_{vmcg}}{V}\frac{(B+1)}{\left[1+\left(\frac{\epsilon}{V}\right)^{-(B+1)(\alpha+1)}\right]} \times \frac{\left[\alpha+B+\alpha B+B\left(\frac{\epsilon}{V}\right)^{(B+1)(\alpha+1)}\right]}{\left[1-B\left(\frac{\epsilon}{V}\right)^{(B+1)(\alpha+1)}\right]} \tag{30}$$

This equation is identical with the modified Chaplygin gas (MCG) model another work of Santos [54].

(iii) For $\xi_0 = 0$, $\alpha = -1/2$, but $A \neq 0$ and $B \neq 0$, then the equation (28) reduces to

$$\left(\frac{\partial p_{vmcg}}{\partial V}\right)_S = \frac{p_{vmcg}}{2V}\frac{(B+1)}{\left[1+\left(\frac{\epsilon}{V}\right)^{-\frac{B+1}{2}}\right]}\frac{\left[B-1+2B\left(\frac{\epsilon}{V}\right)^{\frac{B+1}{2}}\right]}{\left[1-B\left(\frac{\epsilon}{V}\right)^{\frac{B+1}{2}}\right]} \tag{31}$$

It also agrees with the previous work of Thakur [61].

From equation (28), when $\left(\frac{\partial p_{vmcg}}{\partial V}\right)_S < 0$ and α > 0, ξ0 should be negative during the evolution of the universe. We have plot of $\left(\frac{\partial p_{vmcg}}{\partial V}\right)_S$ versus V for various cosmological models as shown in Fig.6, which indicates that the negative adiabatic condition of the fluid leads to stable. So, we have seen that when ξ0 < 0, our model is stable during the universe's evolution.

Using the thermodynamics relation, we also verified that the specific heat was positive at constant volume. The specific heat can be expressed in terms of temperature and entropy as

$$c_V = T\left(\frac{\partial S}{\partial T}\right)_V = \left(\frac{\partial U}{\partial T}\right)_V = V\left(\frac{\partial \rho_{vmcg}}{\partial T}\right)_V \tag{32}$$

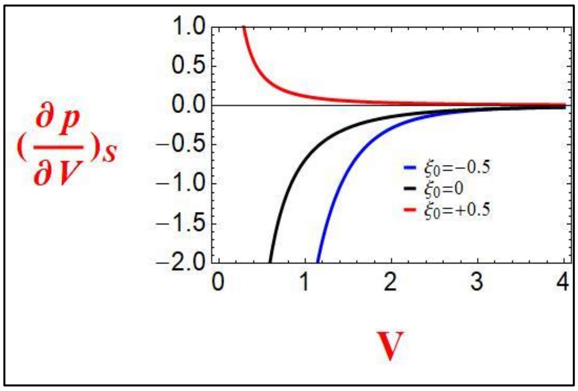
where T and S are represents temperature and entropy, respectively. The equation is utilized to determine the fluid's temperature, T = $\left(\frac{\partial U}{\partial S}\right)_V$ [70], it can be expressed

$$T = \left(\frac{\partial U}{\partial b}\right)_V\left(\frac{\partial b}{\partial S}\right)_V \tag{33}$$

Using equation (9), the expression temperature can be written as

$$T = \frac{V^{-[(1+B-\sqrt{3}\xi_0)(\alpha+1)-1]}}{(\alpha+1)}\left[\frac{A}{1+B-\sqrt{3}\xi_0}+\frac{b}{V^{(1+B-\sqrt{3}\xi_0)(\alpha+1)}}\right]^{-\frac{\alpha}{\alpha+1}}\left(\frac{\partial b}{\partial S}\right)_V, \tag{34}$$

Since b is a universal constant, we find that $\left(\frac{db}{ds}\right)_V = 0$. This implies that, regardless of the pressure and volume, the temperature is zero. However, in the case of Chaplygin gas, the temperature does change with expansion. Therefore, we must consider $\frac{db}{ds} \neq 0$.



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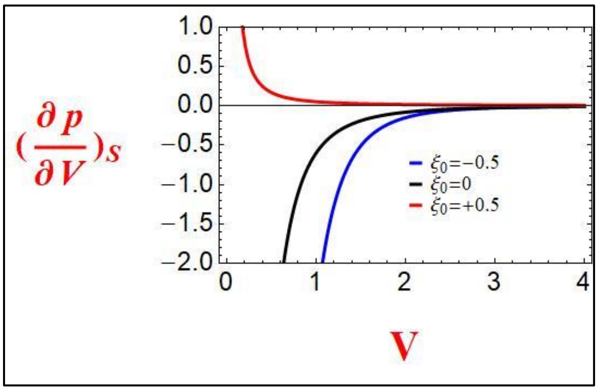

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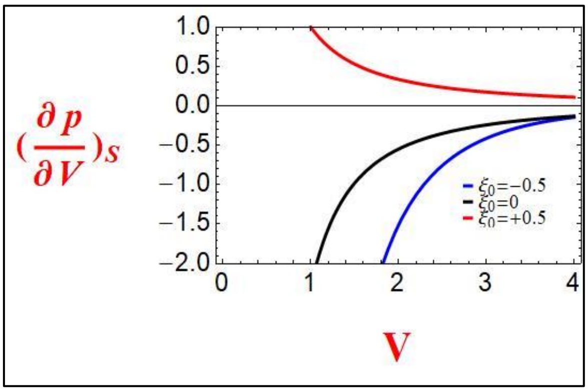

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Fig 6: The plot of pressure gradient $\left(\frac{\partial p}{\partial V}\right)_S$ as a function of volume V in different values of viscous coefficient ξ0, where we have taken for (a) A = 1, B = 1/3, α = 0.5 & b = 1; (b) A = 10, B = 1/3, α = 0.5 & b = 1; (c) A = 1, B = 1/3, α = 0.5 & b = 10, and also clearly shows that ∂pvmcg < 0 (VMCG model) for ξ0 < 0 throughout the evolution period.

Lacking specific information on how b depends on S, we assume that $\frac{db}{ds} > 0$ [57, 58], which allows us to derive positive temperatures that decrease through adiabatic expansion. Equation (9) can be obtained by dimensional analysis

$$[b] = [U]^{\alpha+1}[V^{(B-\sqrt{3}\xi_0)(\alpha+1)}] \tag{35}$$

Since we are aware that [U] = [T· S], the above equation can be expressed as

$$[b] = [T]^{\alpha+1}[S]^{\alpha+1}[V]^{(B-\sqrt{3}\xi_0)(\alpha+1)} \tag{36}$$

Since it is challenging to obtain an analytical solution of *b* from equation (36), we choose an empirical expression of *b* as a trial case and then determine whether the ensuing expressions fulfill standard relations of thermodynamics. However, as *b* is only dependent on entropy, its expression will be

$$b = \left(\tau v^{B-\sqrt{3}\xi_0}\right)^{\alpha+1} S^{\alpha+1} \tag{37}$$

In this case, *v* represents the volume dimension and *τ* (the constant) represents merely the temperature. Presently, we get

$$\left(\frac{db}{dS}\right)_V = (\alpha+1)\left(\tau v^{B-\sqrt{3}\xi_0}\right)^{\alpha+1} S^{\alpha} \tag{38}$$

Using equations (37) and (38), the equation (34) becomes

$$T = V^{-\left[(1+B-\sqrt{3}\xi_0)(\alpha+1)-1\right]}\rho^{-\alpha}\left(\tau v^{B-\sqrt{3}\xi_0}\right)^{\alpha+1}S^{\alpha} \tag{39}$$

The expression above can be streamlined to

$$T = \frac{\tau v^{B-\sqrt{3}\xi_0}}{V^{B-\sqrt{3}\xi_0}}\left[1+\left(\frac{V}{\epsilon}\right)^{(1+B-\sqrt{3}\xi_0)(\alpha+1)}\right]^{-\frac{\alpha}{\alpha+1}} \tag{40}$$

Putting the value of b in equation (39), the entropy is given as

$$S = \left(\frac{A}{1+B-\sqrt{3}\xi_0}\right)^{\frac{1}{\alpha+1}} \frac{\left(\frac{V^{1+B-\sqrt{3}\xi_0}}{\tau v^{B-\sqrt{3}\xi_0}}\right)}{\left[\left(\frac{\tau v^{B-\sqrt{3}\xi_0}}{TV^{B-\sqrt{3}\xi_0}}\right)^{\frac{\alpha+1}{\alpha}}-1\right]^{\frac{1}{\alpha+1}}} \tag{41}$$

which means that it may also be written as

$$S = \left(\frac{A}{1+B-\sqrt{3}\xi_0}\right)^{\frac{1}{\alpha+1}}\left(\frac{V}{T}\right) \frac{\left(\frac{TV^{B-\sqrt{3}\xi_0}}{\tau v^{B-\sqrt{3}\xi_0}}\right)^{\frac{\alpha+1}{\alpha}}}{\left[1-\left(\frac{TV^{B-\sqrt{3}\xi_0}}{\tau v^{B-\sqrt{3}\xi_0}}\right)^{\frac{\alpha+1}{\alpha}}\right]^{\frac{1}{\alpha+1}}}. \tag{42}$$

From equation (42), we see that the condition $0 < TV^{(B-\sqrt{3}\xi_0)} < \tau v^{(B-\sqrt{3}\xi_0)}$ must hold for positive and finite entropy. This condition is confirmed as it meets the constraints $\tau > T > 0$, and $v < V < \infty$, where v represents the lowest volume and τ represents the highest temperature. If $T \to 0$, then equation (42) yields S = 0, indicating that the thermodynamics' third law is satisfied. Substituting equations (40) and (42) into (32), it follows that

$$c_V = T\left(\frac{\partial S}{\partial T}\right)_V = \frac{1}{\alpha}\left(\frac{A}{1+B-\sqrt{3}\xi_0}\right)^{\frac{1}{\alpha+1}}\left(\frac{V}{T}\right) \frac{\left(\frac{TV^{B-\sqrt{3}\xi_0}}{\tau v^{B-\sqrt{3}\xi_0}}\right)^{\frac{\alpha+1}{\alpha}}}{\left[1-\left(\frac{TV^{B-\sqrt{3}\xi_0}}{\tau v^{B-\sqrt{3}\xi_0}}\right)^{\frac{\alpha+1}{\alpha}}\right]^{\frac{\alpha+2}{\alpha+1}}} \tag{43}$$

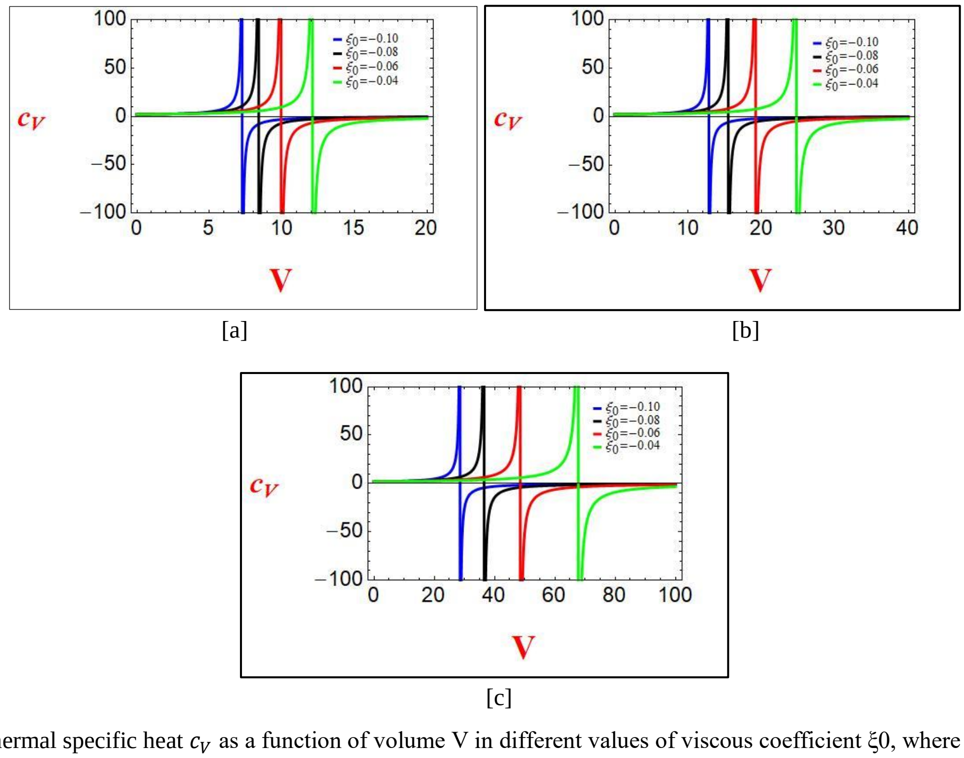


Fig 7: The plot of thermal specific heat c_V as a function of volume V in different values of viscous coefficient ξ0, where we have taken for (a) A = 1, B = 1/3, T = 1, τ = 2.73, S = 1, v = 1 & α = 0.5; (b) A = 1, B = 1/3, T = 0.75, τ = 2.73, S = 1, v = 1 & α = 0.5; (c) A = 1, B = 1/3, T = 0.50, τ = 2.73, S = 1, v = 1 & α = 0.5.

In terms of entropy S, the aforementioned equation can also be written as

$$c_V = \frac{1}{\alpha}\frac{S}{\left[1-\left(\frac{TV^{B-\sqrt{3}\xi_0}}{\tau v^{B-\sqrt{3}\xi_0}}\right)^{\frac{\alpha+1}{\alpha}}\right]} \quad (44)$$

When 0 < $TV^{(B-\sqrt{3}\xi_0)} < \tau v^{(B-\sqrt{3}\xi_0)}$ i.e., τ > T > 0 and v < V < ∞ and α > 0, then c_V is positive and c_V > 0, always satisfied. We found that in this model, the specific heat capacity disappears at zero temperature, or cV 0, for T 0; verifies the third law of thermodynamics' validity. Thus, given the identical circumstance, we also saw that the thermal heat capacity cV and the entropy S have positive values.
Thus, for values of ξ0 < 0 and α = 1/2 during the evolution, the VMCG model is thermodynamically stable. The plot of specific heat vs volume for a given value of α = 1/2 and different values of ξ0 is displayed in Fig.7. In equation (44), if we put ξ0 = 0, we obtain an expression that is comparable to c_V, which was discovered by Santos [54] and correlates to MCG.
Using equations (9), (37) and (42) to calculate the internal energy of this model, we get

$$U = V\left(\frac{A}{1+B-\sqrt{3}\xi_0}\right)^{\frac{1}{\alpha+1}}\left[1-\frac{1}{1-\left(\frac{\tau v^{B-\sqrt{3}\xi_0}}{TV^{B-\sqrt{3}\xi_0}}\right)^{\frac{\alpha+1}{\alpha}}}\right]^{\frac{1}{\alpha+1}} \quad (45)$$

For the stability of the VMCG model, we looked at the isothermal condition, or $\left(\frac{\partial p}{\partial V}\right)_T < 0$. Applying $p = p\ (V,\ T)$ from thermodynamic relations and resolving equations (14) and (37), we obtain

$$p_{vmcg} = -\left(\frac{A}{1+B-\sqrt{3}\xi_0}\right)^{\frac{1}{\alpha+1}}\frac{\left[(1+B-\sqrt{3}\xi_0)-\left(\frac{\tau v^{B-\sqrt{3}\xi_0}}{TV^{B-\sqrt{3}\xi_0}}\right)^{\frac{\alpha+1}{\alpha}}\right]}{\left(\frac{\tau v^{B-\sqrt{3}\xi_0}}{TV^{B-\sqrt{3}\xi_0}}\right)\left[\left(\frac{\tau v^{B-\sqrt{3}\xi_0}}{TV^{B-\sqrt{3}\xi_0}}\right)^{\frac{\alpha+1}{\alpha}}-1\right]^{\frac{1}{\alpha+1}}} \quad (46)$$

The aforementioned statement can also be expressed in terms of entropy, therefore we have

$$p_{vmcg} = -\left(\frac{TS}{V}\right)\left[(1+B-\sqrt{3}\xi_0)-\left(\frac{\tau v^{B-\sqrt{3}\xi_0}}{TV^{B-\sqrt{3}\xi_0}}\right)^{\frac{\alpha+1}{\alpha}}\right] \quad (47)$$

where density is expressed in temperature terms by

$$\rho_{vmcg} = \left(\frac{A}{1+B-\sqrt{3}\xi_0}\right)^{\frac{1}{\alpha+1}}\left[1-\frac{1}{1-\left(\frac{\tau v^{B-\sqrt{3}\xi_0}}{TV^{B-\sqrt{3}\xi_0}}\right)^{\frac{\alpha+1}{\alpha}}}\right]^{\frac{1}{\alpha+1}} \quad (48)$$

The form of the relevant thermal EoS parameter of VMCG is

$$\omega_{vmcg} = \frac{\left[(1+B-\sqrt{3}\xi_0)-\left(\frac{\tau v^{B-\sqrt{3}\xi_0}}{TV^{B-\sqrt{3}\xi_0}}\right)^{\frac{\alpha+1}{\alpha}}\right]}{\left(\frac{\tau v^{B-\sqrt{3}\xi_0}}{TV^{B-\sqrt{3}\xi_0}}\right)^{\frac{\alpha+1}{\alpha}}} \quad (49)$$

Temperature (T) also affects the expression above. When the temperature is very high in the early stages of the universe, T → τ, the equation (49) indicates that the cosmos is dominated by dust, as ω_{vmcg} = 0 and p_{vmcg} = 0. The ΛCDM model is shown by the equation (49), which yields ω_{vmcg} = —1 when the universe is in its late stage, when the temperature is extremely low, i.e., T → 0.
We will now analyze equation (46) to see if $\left(\frac{\partial p_{vmcg}}{\partial V}\right)_T \leq 0$. Consequently, we obtain

$$\left(\frac{dp_{vmcg}}{dV}\right)_T = \left(\frac{1}{V}\right)\left(\frac{A}{1+B-\sqrt{3}\xi_0}\right)^{\frac{1}{1+\alpha}} \frac{\left(1-N^{-\frac{1+\alpha}{\alpha}}\right)^{-\frac{1+\alpha}{\alpha}}}{N^{\frac{1+\alpha}{\alpha}}\left(1+N^{\frac{1+\alpha}{\alpha}}\right)}$$
$$\left[B^2\left[1-(1+\frac{1}{\alpha})N^{\frac{1+\alpha}{\alpha}}\right]-\xi_0\left[\sqrt{3}-3\xi_0+(3\xi_0-\sqrt{3}+\frac{3\xi_0}{\alpha})N^{\frac{1+\alpha}{\alpha}}\right]\right.$$
$$\left.+B\left[1-2\sqrt{3}\xi_0+(2\sqrt{3}\xi_0-1+\frac{2\sqrt{3}\xi_0}{\alpha})N^{\frac{1+\alpha}{\alpha}}\right]\right], \quad (50)$$

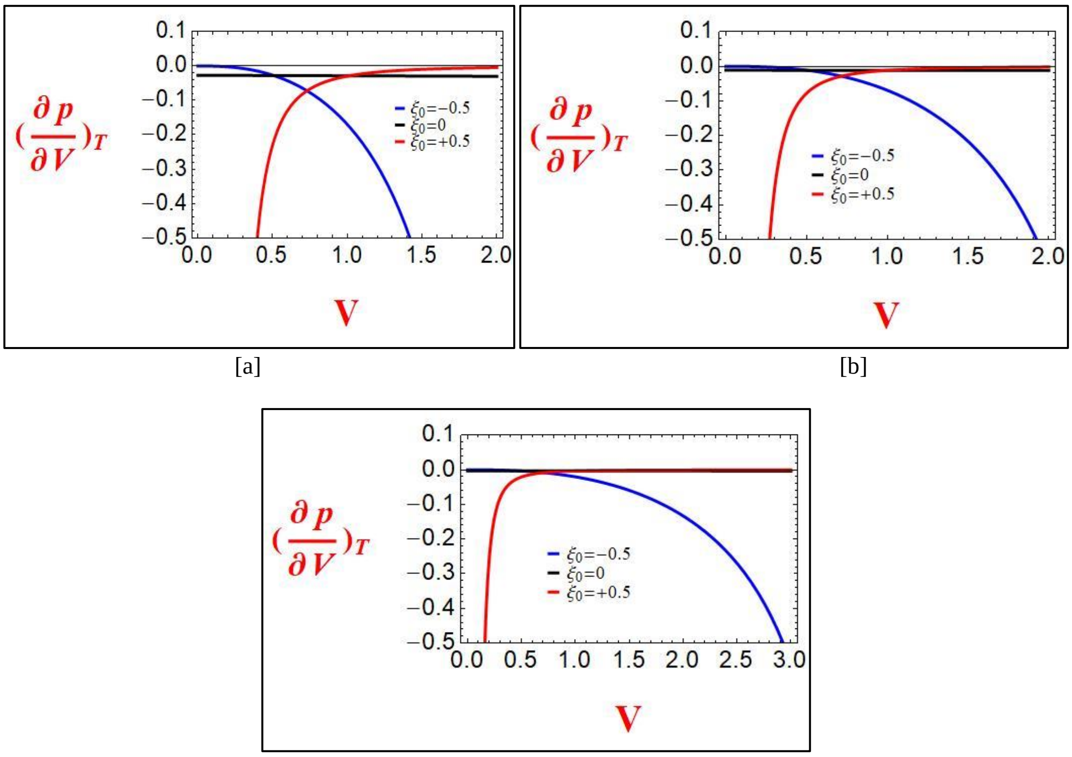
This clearly in Fig.8 indicates that the value of ξ0 must be negative for $\left(\frac{\partial p_{vmcg}}{\partial V}\right)_T < 0$ throughout the evolution. Notably, when ξ0 equals zero, $\left(\frac{\partial p_{vmcg}}{\partial V}\right)_T$ becomes zero as well. At this point, we take a moment to compare our findings with the earlier work of Santos *et al*. [53, 54]. They only calculated pressure as a function of temperature in their analysis of the generalized Chaplygin gas model, and they applied the same hypothesis to the MCG model, which led to $\left(\frac{\partial p_{vmcg}}{\partial V}\right)_T = 0$ in both scenarios.
However, in our analysis, with ξ0 > 0, $\left(\frac{\partial p_{vmcg}}{\partial V}\right)_T$ is consistently less than zero. This means that the isobaric curves in our viscous generalized Chaplygin gas model do not overlap with the isotherms within the thermodynamic state diagram. This distinction represents a meaningful advancement in our analysis.
Thus, we conclude that $\left(\frac{\partial p_{vmcg}}{\partial V}\right)_S < 0$ and $\left(\frac{\partial p_{vmcg}}{\partial V}\right)_T < 0$ for negative value of ξ0,

4. Pressure & Volume

Using the following extreme conditions, we have investigated the relationship between effective pressure and volume.
(i) For a small volume $V \ll \epsilon$, the effective pressure $p_{eff} \approx (B-\sqrt{3}\xi_0)\rho_{vmcg}$, since $\frac{\epsilon}{V} = \infty$. Both pressure and energy density are extremely high in this instance. Equation (40) can now be used to express the temperature as

$$T = \frac{\tau v^{B-\sqrt{3}\xi_0}}{V^{B-\sqrt{3}\xi_0}} \quad (51)$$

Early in the universe's existence, the transition from V to v (lowest volume) is associated with the transition from T to τ (maximum temperature). At this point, the temperature is significantly high. By applying equation (51) and utilizing the



[a] [b] [c]

Fig 8: The plot of pressure gradient $\left(\frac{\partial p}{\partial V}\right)_T$ as a function of volume V in different values of viscous coefficient ξ0, where we have taken for (a) A = 1, B = 1/3, T = 1, τ = 2.73, S = 1, v = 1 & α = 0.5; (b) A = 1, B = 1/3, T = 0.75, τ = 2.73, S = 1, v = 1 & α = 0.5; (c) A = 1, B = 1/3, T = 0.50, τ = 2.73, S = 1, v = 1 & α = 0.5.

Relationship $\rho_{vmcg} = \frac{TS}{V}$, we derive

$$\rho_{vmcg} = \frac{S\tau v^{B-\sqrt{3}\xi_0}}{V^{1+B-\sqrt{3}\xi_0}} \tag{52}$$

Hence,

$$UV^{B-\sqrt{3}\xi_0} = \rho_{vmcg} V^{1+B-\sqrt{3}\xi_0} = S\tau v^{B-\sqrt{3}\xi_0} \tag{53}$$

At the early universe, we know $p_{eff} \approx (B - \sqrt{3}\xi_0)\rho_{vmcg}$, i.e., $p_{eff}V \approx (B - \sqrt{3}\xi_0)$U, we have $p_{eff}V^{1+B-\sqrt{3}\xi_0} = S\tau v^{B-\sqrt{3}\xi_0}$ (using equation (53)). In an adiabatic process, since the entropy is constant, the relation leads to $p_{eff}V^{1+B-\sqrt{3}\xi_0} = S\tau v^{B-\sqrt{3}\xi_0} =$ constant. Thus, for small volume, such as, at high temperature, the VMCG model is found to act as a fluid of $\gamma = \frac{c_p}{c_v} = (1 + B - \sqrt{3}\xi_0)$. The EoS can also be rewritten as $p_{eff} = (\gamma - 1)\,\rho_{vmcg}$.

Since the pressure-volume relation at early universe is radiation dominated i.e., $B = \frac{1}{3}$, the value of $\gamma = (1 + \frac{1}{3} - \sqrt{3}\xi_0) = 4/3 - \sqrt{3}\xi_0$. So, we get pressure-volume relation $p_{eff}V^{\frac{4}{3}} = constant$ for $\xi_0 = 0$; $p_{eff}V^1 = constant$ for $\xi_0 = \frac{1}{3\sqrt{3}}$; $p_{eff}V^2 = constant$ for $\xi_0 = -\frac{2}{3\sqrt{3}}$ and $p_{eff}V^{\frac{2}{3}} = constant$ for $\xi_0 = \frac{2}{3\sqrt{3}}$. Thus, this model behaves like a photon gas. The equation of adiabatic photon gas coincides with extreme relativistic electron gas.

(ii) For large volume ($V \gg \epsilon$), so we have $\frac{\epsilon}{V} = 0$. The entropy is significantly lower at low temperatures because of the low density at this point; thus, we obtain

$$p_{eff} = -\rho_{vmcg} \tag{54}$$

From equation (12), in this case density becomes

$$\rho_{vmcg} \approx \left(\frac{A}{1 + B - \sqrt{3}\xi_0}\right)^{\frac{1}{\alpha+1}} \tag{55}$$

Now using equations (54) and (55), we get

$$p_{eff} = -\left(\frac{A}{1 + B - \sqrt{3}\xi_0}\right)^{\frac{1}{\alpha+1}} \tag{56}$$

Dark energy is the state in which the effective pressure is negative and constant for adiabatic systems, which occur when the universe is late in its life cycle and at a lower temperature.

5. Conclusions

This work has investigated the thermodynamic stability and positivity of the thermal heat capacity (c_V) of the VMCG model in the context of the FRW universe. We also examined the behavior of an expanding universe using physical properties such as effective pressure, effective EoS, deceleration parameter, and sound speed. An overview of the previously mentioned parameters is provided below:

1. The energy density of viscous modified Chaplygin gas (VMCG), which is affected by bulk viscosity, decreases with increasing volume and functions as a single dark fluid. As shown in Fig.1, when volume is large, the energy density drops from a high starting value to a lower, with viscosity frequently raising the total energy density during evolution.
2. Using different values of the viscous parameter ξ_0 to plot effective pressure has produced outcomes. We also discover that, as shown in Fig.2, the pressure can be either positive or negative, depending on the value of ξ_0. Furthermore, for large volumes, we observed that this model acts as a dust-dominated universe, while at small volumes, it displays a constant negative pressure. For a Chaplygin-type gas, $\xi_0 = 0$ and $\xi_0 > 0$, the pressure is always negative.
3. The EoS parameter for the VMCG model shows that the viscous parameter ξ_0 governs the decelerated and accelerated phases of our universe, as shown in Fig.3.
4. As seen in Fig.4, the deceleration parameter q usually begins positively (deceleration) and shifts to negative values as the volume grows in a universe dominated by viscous modified Chaplygin gas, suggesting a seamless transition from early deceleration to late-time accelerating expansion. The universe is generally driven into a de Sitter phase by the viscous effect ($q \rightarrow -1$). For negative values of ξ_0, the universe decelerates when $V < V_f$, and accelerates when $V > V_f$. When the flip

occurs, we calculate the flip volume at acceleration. We also show how this flip volume causes the cosmos to accelerate at larger volumes while decelerating at smaller ones.

5. Sound speed varies with volume for a variety of ξ0 values, as shown in Fig.5. We investigated the stability of the VMCG model using adiabatic sound speed in the range of $-\frac{1}{3\sqrt{3}} < \xi_0 < \frac{1}{3\sqrt{3}}$ and found that it was stable for the value of $\alpha = \frac{1}{2}, A > 0, B > 0$. For $\xi_0 \leq 0$, the VMCG model [58] always has a positive sound speed, but the GCCG model does not have stable areas at the late universe [70].
6. Also, we studied the specific heat, adiabatic states, and isothermal states in relation to the thermodynamic stability requirement. We show that the VMCG model expands adiabatically for any volume and that the expansion is thermodynamically stable at any pressure, as shown in Fig.6. We also confirmed that the third law of thermodynamics is true. For the GCG and MCG models, the specific heat at constant volume is always non-negative [53, 54]; however, for the VMCG model, it is always positive when $0 < T < \tau$, $v < V < \infty$, and $\alpha > 0$. Furthermore, we found that at temperature $T = 0$, entropy is zero, demonstrating that the VMCG model conforms to the third law of thermodynamics. In adiabatic conditions, the VMCG model is found to be thermodynamically stable for $\xi 0 < 0$.
7. The VMCG model is thermodynamically stable because the specific heat at constant volume (c_V) remains positive during the expansion as shown in Fig.7. In the early universe, the specific heat at constant volume decreases to zero when the volume V is very small, and the value of (c_V) stays positive as the volume of the late universe expands. The evolution of the viscous modified Chaplygin gas is frequently investigated to reveal a change from a phase of deceleration (small volume) to one of rapid expansion (large volume), while (c_V) stays positive.
8. Finally, equation (48) makes it evident that $\left(\frac{\partial p_{vmcg}}{\partial V}\right)_T$ remains negative ever, when $\xi 0 = 0$, $\left(\frac{\partial p_{vmcg}}{\partial V}\right)_T$ is zero. Santos *et al.* [53, 54] assumed $\left(\frac{\partial p_{vmcg}}{\partial V}\right)_T = 0$ in order to derive the expression for b. This is similar to the case they looked at. Specifically, the GCG model solely computed pressure as a function of temperature, and the MCG model was based on the same assumption, that $\left(\frac{\partial p_{vmcg}}{\partial V}\right)_T = 0$ in both cases. However, for $\xi 0 < 0$ in our case, $\left(\frac{\partial p_{vmcg}}{\partial V}\right)_T$ naturally becomes negative, indicating that our viscous modified Chaplygin gas (VMCG) model's isobaric curves do not match its isotherms on the diagram of thermodynamic states.

This model based on theoretical findings, has been used to study the thermodynamic nature of the universe.

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7. Declarations

Conflicts of Interest The authors declare that they have no competing interests.

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