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Exponential Stability Analysis of a Gradient-Flow Model for Haar Wavelet Coefficients in Signal Processing

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Abstract
This paper investigates the dynamical modeling of Haar wavelet coefficient estimation in signal processing. The problem is formulated as the minimization of a strictly convex quadratic energy functional composed of data-fidelity and regularization terms. By applying continuous-time Euclidean gradient flow, the estimation problem is transformed into a linear first-order dynamical system. Under the assumption that the system matrix M+W is symmetric positive definite, spectral analysis and Lyapunov theory are employed to establish the existence of a unique equilibrium point and to prove global exponential convergence toward the minimizer. The exponential decay rate is explicitly characterized by the smallest eigenvalue of the system matrix. Numerical experiments on both synthetic and ECG signals validate the theoretical analysis and demonstrate stable and robust convergence behavior. The proposed framework provides a rigorous connection between wavelet-based coefficient estimation and dynamical systems stability theory.

Keywords: Haar wavelet, gradient-flow dynamics, exponential stability, Lyapunov method

1. Introduction

Wavelet analysis constitutes a fundamental mathematical framework in modern signal processing due to its multiresolution structure and localized representation properties [8,12]. Among various wavelet families, the Haar wavelet remains particularly attractive because of its orthogonality, piecewise constant structure, and computational efficiency, which enable practical implementation in large-scale applications [14]. In biomedical signal processing, especially in electrocardiogram (ECG) analysis, wavelet coefficients play a central role in denoising, reconstruction, and feature extraction [10,13].

From a variational viewpoint, coefficient estimation problems can be formulated as the minimization of quadratic energy functionals consisting of data-fidelity and regularization terms, a structure widely encountered in imaging and inverse problems [4,5]. Continuous optimization theory provides rigorous analytical tools for studying such models [6], and gradient flows associated with strongly convex energies are known to exhibit well-structured convergence behavior [2].

However, despite the extensive use of Haar-based transforms, coefficient computation is typically treated as a static algebraic procedure. A systematic formulation of Haar wavelet coefficient estimation as a continuous-time gradient-flow dynamical system, together with a rigorous spectral and Lyapunov stability analysis providing explicit exponential convergence rates, has not been thoroughly investigated.

Motivated by this observation, the present study formulates Haar coefficient estimation as the minimization of a strictly convex quadratic energy functional and derives the associated continuous-time gradient-flow model in the Euclidean setting. Particular attention is devoted to identifying structural conditions under which the resulting linear dynamical system admits a unique equilibrium point and achieves global exponential stability. In particular, the symmetric positive definiteness of the system matrix arising from the orthonormal Haar basis and the regularization operator is shown to guarantee both strict convexity of the energy functional and quantitative exponential convergence.

The analysis relies on tools from convex analysis, spectral theory of symmetric positive definite matrices, and Lyapunov stability theory for linear systems [1,9]. By combining eigenvalue analysis with a quadratic Lyapunov function, explicit decay estimates are derived, establishing a direct relationship between the convergence rate and the spectral properties of the system matrix.

2. Preliminaries

In this section, we recall the fundamental concepts and analytical tools that are directly employed in the modeling and stability analysis of the proposed gradient-flow system. The presentation is structured to reflect the logical chain underlying the main results: spectral properties of symmetric positive definite matrices, strict convexity of quadratic functionals, gradient-flow dynamics, and exponential stability via spectral and Lyapunov methods

2.1. Symmetric Positive Definite Matrices and Spectral Bounds

Let $M \in R^{n\times n}$. The matrix M is said to be **symmetric positive definite (SPD)** if it satisfies $M = M^T$ and $x^T Mx > 0$. For all $x \neq 0$ If M is SPD then all its eigenvalues are real and strictly positive. Denoting the smallest largest eigenvalues by $\lambda_{\min}(M) > 0$ and $\lambda_{\max}(M) > 0$ respectively, the following spectral bound hold: $\lambda_{\min}(M)\|x\|^2 \le x^T Mx \le \lambda_{\max}(M)\|x\|^2, \forall x \in \mathbb{R}^n.$

In particular, $\lambda_{\min}(M) > 0$, which implies that M is invertible. These spectral inequalities play a central role in deriving quantitative exponential stability estimates in the Lyapunov analysis of linear systems [9].

2.2. Strict Convexity of Quadratic Functionals

Let $E:\mathbb{R}^n \to \mathbb{R}$ be a twice continuously differentiable function. The function E is said to **strictly convex** if $E(\theta x + (1-\theta)y) < \theta E(x) + (1-\theta)E(y)$ for all $x \neq y$ and $\theta \in (0,1)$ Consider a quadratic functionals of the form $E(a) = \frac{1}{2} aMa^T - a^T d$, strict convexity is equivalent to the Hessian matrix M being symmetric positive definite. In that case, the functional admits a unique global minimizer, characterized by the condition $\nabla E(a^*) = 0$.

This property follows from standard results in convex analysis and gradient-flow theory [1,2]. In the present work, this result is applied to the quadratic energy functional whose Hessian is given by M+W.

2.3. Euclidean Gradient Flow

Given a differentiable functional $E:\mathbb{R}^n \to \mathbb{R}$, the associated Euclidean gradient flow system is defined by $a^{\cdot}(t) = -\gamma \nabla E(a(t)), \gamma > 0$, where γ is a positive Scalar parameter governing the descent rate. Along trajectories of this system, the energy is non-increasing

$$\frac{d}{dt} E\big(a(t)\big) = \nabla E\big(a(t)\big)^T \dot{a}(t) = -\gamma \left\|\nabla E\big(a(t)\big)\right\|^2 \le 0$$

If E is strongly convex, the associated gradient flow generates a linear time-invariant dynamical system with structured spectral properties. In the quadratic case, substituting the expression for ∇E yields a first-order linear differential equation whose stability properties can be analyzed through spectral and Lyapunov techniques [2].

2.4. Exponential Stability of Linear Systems

Consider a linear time-invariant system $x^{\cdot}(t) = Ax(t), \quad x(0) = x_0 \in \mathbb{R}^n$, Where $A \in \mathbb{R}^{n\times n}$. The equilibrium point x=0 is said to be **globally exponentially stable** if there exist constants C>0 and μ>0 such that $\|x(t)\| \le Ce^{-\mu t}\|x(0)\|$ for all $t \ge 0$.

In finite-dimensional spaces, exponential stability is equivalent to the condition that all eigenvalues of A have strictly negative real parts [9]. This spectral characterization provides a direct tool for stability verification of linear gradient-flow systems arising from quadratic energies.

2.5. Quadratic Lyapunov Functions and Grönwall's Inequality

Let $M \in \mathbb{R}^{n\times n}$ be symmetric positive definite and define a quadratic Lyapunov function $V(x) = \frac{1}{2} x^T Mx.$ Then V is positive definite and radially unbounded. Along trajectories of a differentiable system $\dot{x}(t) = f\big(x(t)\big)$, its time derivative is given by $\frac{d}{dt} V\big(x(t)\big) = x(t)^T M\dot{x}(t).$ If one can establish an inequality of the form $\dot{V}(x) \le -\mu V(x)$ for μ>0, then by **Grönwall's inequality** implies $V\big(x(t)\big) \le V\big(x(0)\big)e^{-\mu t}$, which yields exponential decay of the state variable. This analytical mechanism constitutes the key tool used later to derive explicit exponential convergence rates for the gradient-flow model [9].

3. Analytical Results

In this section, we present the main analytical results of the paper. The study proceeds from the explicit construction of the quadratic energy functional associated with Haar approximation to the derivation of its gradient structure and the corresponding gradient-flow dynamical model. We then establish the existence and uniqueness of the equilibrium point, derive the closed-form solution of the linear system, and prove global exponential stability using both spectral arguments and Lyapunov analysis. Finally, explicit convergence rates are obtained in terms of the spectral properties of the system matrix, thereby providing a complete dynamical characterization of the Haar coefficient evolution.

3.1. Haar Approximation Model

Let $f \in \mathbb{R}^n$ denote the discrete representation of a signal defined on a finite interval. $\{\psi_j\}_{j=0}^{N-1}$ be a finite Haar basis, and let $\Psi \in \mathbb{R}^{n\times N}$ denote the corresponding Haar matrix whose columns are the basis vectors.The approximation of the signal is defined by $\hat{f} = \Psi a$, where $a(t) = [a_0(t), a_1(t), \cdots a_{N-1}(t)]^T \in \mathbb{R}^N$ is the coefficient vector. The objective is to determine the coefficient vector aaa such that the L^2 approximation error is minimized [12].

3.2. Quadratic Energy Functional

To determine the optimal coefficients, we introduce the quadratic energy functional $E(a) = \frac{1}{2}\| f - \Psi a\|^2 + \frac{1}{2} aWa^T$, where

- $\Psi = [\psi_0, \psi_1, \psi_2, ..., \psi_{N-1}]$ is the matrix of Haar basis functions?
- $W = diag(w_1, w_2, \cdots, w_{N-1}) \ge 0$ is a weighting (regularization) matrix,
- the first term represents the approximation error,
- the second term provides regularization and improves conditioning.

Proposition 3.1 (Quadratic Representation of the Energy)

The functional E can be written in quadratic form as $E(a) = \frac{1}{2} a^T (M + W) a - d^T a + \frac{1}{2} f^T f$.

Proof: Expanding the squared norm yields $\| f - \Psi a\|^2 = \frac{1}{2}(f - \Psi a)^T (f - \Psi a)$. Therefore, $\| f - \Psi a \|^2 = f^T f - 2a^T \Psi^T f + a^T \Psi^T \Psi a$.

Defining $M = \Psi^T \Psi$ and $d = \Psi^T f$ gives $E(a) = \frac{1}{2} a^T Ma - d^T a + \frac{1}{2} f^T f + \frac{1}{2} a^T Wa,$ which yields the stated form.

Remark 3.1 (Orthonormal Case)

If the Haar basis is complete and orthonormal, then $\Psi^T \Psi = I_N$, and hence $M = I_N$.

In reduced representations, M remains symmetric positive definite but is not necessarily the identity matrix.

3.3. Gradient Structure

We now compute the gradient of the energy functional.

Proposition 3.2 (Gradient of the Energy)

The gradient of E is given by $\nabla E(a) = (M + W)a - d$.

Proof: Since $E(a) = \frac{1}{2} a^T (M + W) a - d^T a + const$, and M+W is symmetric, differentiation yields $\nabla E(a) = (M + W)a - d$. ■

3.4. Gradient-Flow Dynamical System

We now formulate the coefficient estimation as a continuous-time gradient-flow system: $a^{\cdot}(t) = -\gamma \nabla E(a(t)), \gamma > 0$.

Substituting the expression for the gradient gives the linear system $a^{\cdot}(t) = -\gamma(M + W)a(t) + \gamma d$.

Proposition 3.3 (Equilibrium Point)

Assume that M+W is symmetric positive definite. Then the dynamical system admits a unique equilibrium point given by $a^* = (M + W)^{-1} d$.

Proof: The equilibrium satisfies $-\gamma(M + W)a^* + \gamma d = 0$ which yields $(M + W)a^* = d.$ Since M+W is invertible, the solution is unique.

3.5. Structural and Spectral Role of the Regularization Matrix

The regularization matrix W plays a fundamental structural role in both the variational formulation and the dynamical stability of the gradient–flow system.

First, from a variational perspective, the addition of W≥0. guarantees that the matrix M+W remains symmetric positive definite. This ensures strict convexity of the quadratic energy functional and consequently the existence of a unique global minimizer. In particular, if M is already symmetric positive definite, the presence of W reinforces convexity and prevents potential degeneracy in reduced representations.

Second, from a numerical and spectral standpoint, W improves the conditioning of the system matrix. Since the eigenvalues of M+W satisfy $\lambda_i(M + W) \ge \lambda_i(M),$ the minimal eigenvalue increases when W≠0. This spectral shift enhances robustness with respect to perturbations and reduces sensitivity to ill-conditioning.

Third, and most importantly for the dynamical interpretation, the convergence rate of the gradient–flow system is directly determined by the smallest eigenvalue of M+W. As established in Theorem 3.1, the exponential decay rate equals $\gamma\lambda_{min}(M + W).$ Therefore, adding W≥0 shifts the spectrum rightward and increases the exponential convergence rate of the system.

In the orthonormal Haar case where M=I, the system matrix becomes $A = -\gamma(I + W).$

Since all eigenvalues of I+W are strictly positive, the eigenvalues of A lie strictly in the open left half-plane. Consequently, the system is globally exponentially stable, and the rate of decay increases monotonically with the magnitude of W.

Thus, the regularization matrix influences the model at three interconnected levels: convexity of the energy functional, conditioning of the algebraic system, and exponential stability of the associated dynamical flow.

3.6. Explicit Solution of the Gradient–Flow System

Consider the linear dynamical system $a^{\cdot}(t) = -\gamma(M + W)a(t) + \gamma d, \gamma > 0$, where M+W is symmetric positive definite.

Define $A := -\gamma(M + W).$

Since M+W is symmetric positive definite, all eigenvalues of A are strictly negative. Hence, the system is a stable linear time-invariant system.

Proposition 3.4 (Closed-Form Solution)

For any initial condition $a(0) = a_0$, the unique solution is

$a(t) = a^* + e^{At}(a_0 - a^*)$, where $a^* = (M+W)^{-1}d$.

Proof: Define the error variable $e(t) = a(t) - a^*$. Using the equilibrium relation $(M+W)a^* = d$, the dynamics reduce to $\dot{e}(t) = -\gamma(M+W)e(t) = Ae(t)$. The solution of this homogeneous linear system is $e(t) = e^{At}e(0)$. Therefore, $a(t) = a^* + e^{At}(a_0 - a^*)$.

3.7. Global Exponential Stability

We now quantify the convergence rate toward equilibrium.

Theorem 3.1 (Exponential Convergence Rate)

Assume M+W is symmetric positive definite. Then the equilibrium a* is globally exponentially stable and satisfies $\|a(t) - a^*\| \le e^{-\gamma\lambda_{\min}(M+W)t}\|a_0 - a^*\|$.

Proof **:**Since M+W is symmetric positive definite, it admits the orthogonal diagonalization $M + W = Q\Lambda Q^T$, where Q is orthogonal and $\Lambda = diag(\lambda_1, \ldots, \lambda_N), \quad \lambda_i > 0$. Hence, $A = -\gamma Q\Lambda Q^T$ Therefore, $e^{At} = Qe^{-\gamma\Lambda t}Q^T$. For symmetric matrices, the spectral norm satisfies $\|e^{At}\| = \max_i e^{-\gamma\lambda_i t} = e^{-\gamma\lambda_{\min}(M+W)t}$ Thus, $\|e(t)\| \le e^{-\gamma\lambda_{\min}(M+W)t}\|e(0)\|$.

3.8. Lyapunov Stability Analysis

We now provide an alternative proof that does not rely on explicit spectral decomposition.

Let $e(t) = a(t) - a^*$. Then $\dot{e}(t) = -\gamma(M+W)e(t)$.

Theorem 3.2 (Lyapunov Exponential Stability)

Suppose

- M is symmetric positive definite,
- W is symmetric positive semidefinite,
- γ>0.

Then a* is globally exponentially stable.

Proof:

Consider the quadratic Lyapunov function $V(e) = \frac{1}{2}e^T Me$.

Because M>0, $\frac{\lambda_{\min}(M)}{2}\|e\|^2 \le V(t) \le \frac{\lambda_{\max}(M)}{2}\|e\|^2$.

Differentiating along trajectories:

$\dot{V}(e) = e^T M\dot{e} = -\gamma e^T M(M+W)e.$

Since M>0 and W≥0, the matrix M(M+W) is positive definite. Hence, $\dot{V}(e) \le -\gamma\lambda_{\min}(M(M+W))\|e\|^2$. Using the upper spectral bound on V, $\|e\|^2 \ge \frac{2}{\lambda_{\max}(M)}V(e)$, we obtain $\dot{V}(e) \le -2\mu V(e),$ where $\mu = \frac{\gamma\lambda_{\min}(M(M+W))}{\lambda_{\max}(M)}$.

Applying Grönwall's inequality yields $V(e(t)) \le V(e(0))e^{-2\mu t}$. Therefore, $\|e(t)\| \le Ce^{-\mu t}\|e(0)\|$,for some constant C>0. Thus the equilibrium is globally exponentially stable.

4. Numerical Results and Validation

In this section, a series of numerical experiments on the Haar coefficients are described in order to validate the proposed model.

All simulations were implemented in MATLAB R2023 using double-precision floating-point arithmetic. These implementations are based on the continuous and discretized formulations that were previously obtained in Sections 3 and 4.

For evaluation, two signals are considered:

1. A synthetic deterministic signal.
2. A real electrocardiogram (ECG) signal obtained from the PhysioNet source.

The purpose is to demonstrate both the theoretical correctness of the model and its practical stability against real data.

4.1. Synthetic Signal Experiment

To confirm convergence analytically, the following test signal is examined:

$$f(x) = sin(4\pi x) + 0.5cos(8\pi x)$$

The synthetic test signal is shown in Figure (1).

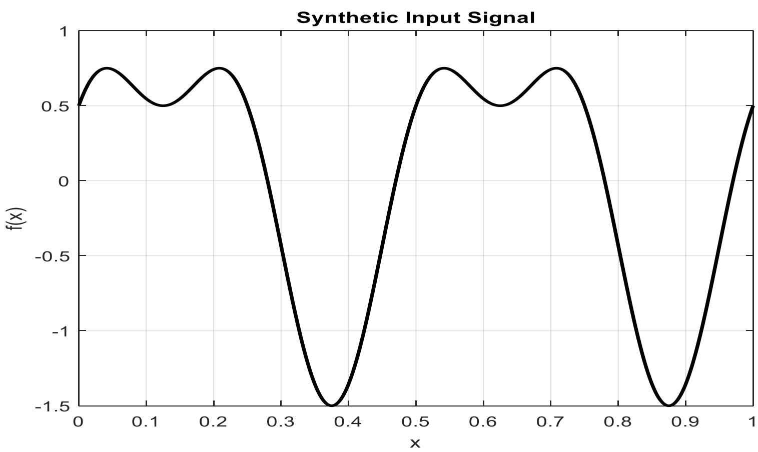

Fig 1:

This signal is defined on the interval $[0,1]$ and is uniformly sampled using $N = 64$ points (Figure (a)).

The orthonormal Haar basis matrix $H^{64\times 64} \in \mathbb{R}$ is formed, and the optimal coefficients are explicitly obtained from the relation $\hat{f} = H^T a$, The dynamic evolution of the coefficients is governed by the full gradient-flow model $\dot{a}(t) = -\gamma(M+W)a(t) + \gamma d$, where $M = \Psi\Psi^T, d = \Psi^T f$.

In the present numerical implementation, a complete orthonormal Haar matrix is used, hence M=I. When no additional regularization is imposed (W=0) [11], the system reduces to

$\dot{a}(t) = -\gamma(a(t) - a^*)$, with equilibrium $a^* = d$. Thus, the numerical experiment corresponds to a special case of the general theoretical model [2].

Figure (b) shows the temporal evolution of selected Haar coefficients. All coefficients converge uniformly toward their optimal values a^* The convergence rate matches the predicted exponential behavior, which confirms the theoretical stability. The observed decay rate is consistent with the theoretical exponential convergence predicted by the stability analysis.

In the general case, the decay rate depends on the spectral properties of the matrix M+W and is given by $e^{-\mu t}$ where

$$\mu = \frac{\gamma\lambda_{\min}\left(M\left(M+W\right)\right)}{\lambda_{\max}\left(M\right)}.$$

In the present numerical experiment, since a complete orthonormal Haar matrix is used (M=I) and no additional regularization is imposed (W=0), the rate reduces to $e^{-\mu t}$, which matches the numerical observations.

To numerically validate stability, the following Lyapunov function is evaluated along the trajectories (Figure (c)):

$$V(t) = \frac{1}{2}\left\| a(t) - a^*(t) \right\|^2$$

Figure (d) shows that $V(t)$ decreases uniformly over time. As $t \to \infty$, $V(t) \to 0$. The decay curve follows the theoretical exponential law, $V(t) = V(0)e^{-\gamma t}$ This provides numerical confirmation of global exponential stability.

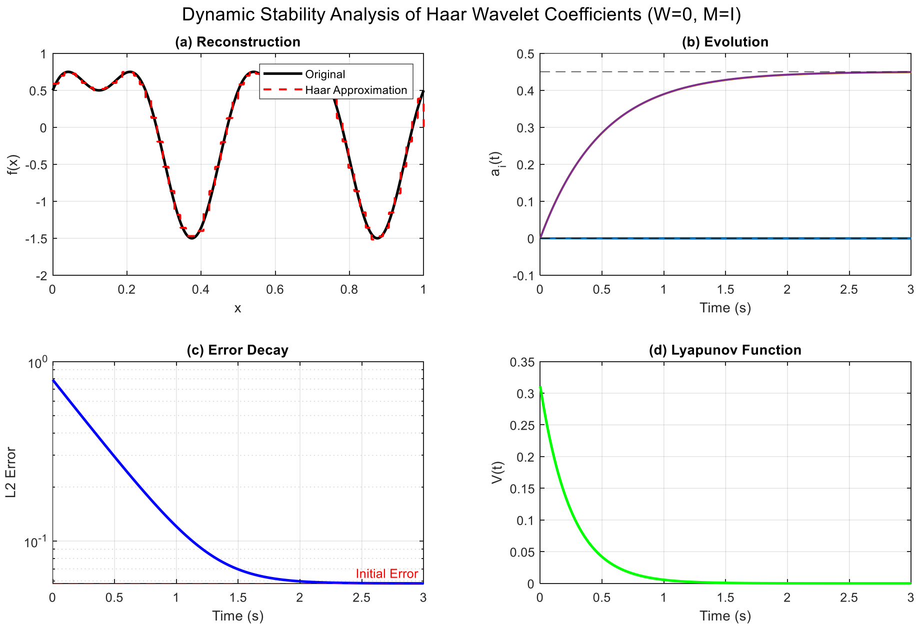

Fig 2:

4.2. Real ECG Signal Experiment

To evaluate the practical applicability of the proposed framework, a real ECG signal obtained from the PhysioNet database was considered. The ECG signal was sampled in discrete form with a sampling fr4.2.equency of 128 Hz. The selected time window covers several seconds, allowing observation of a complete cardiac cycle including its principal components [10].

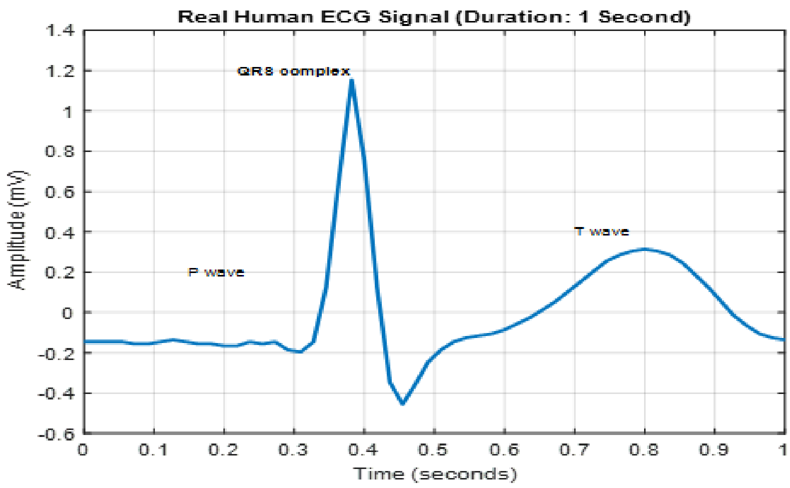

Fig 3:

In this section, the signal reconstruction capability using the proposed dynamical model is examined [13], As shown in Figure (a), the reconstructed signal $f_{rec}(t) = H^T a(t)$ is obtained from the converged coefficients a(t). The reconstructed signal is compared with the original ECG signal. The very high agreement between these two curves confirms the correct implementation of the Haar transform and demonstrates the practical capability of the framework in accurately reconstructing the signal from dynamically evolved coefficients.

Figure (b) illustrates the temporal evolution of selected Haar coefficients for the ECG signal. Despite the piecewise-irregular and nonstationary nature of biomedical ECG data, the coefficients converge smoothly and no unstable oscillatory behavior is observed. This confirms the robustness of the model under realistic signal conditions.

Figure (d) presents the evolution of the Lyapunov function V(t) for the ECG signal case. The function decreases monotonically from its initial value, confirming that the proposed dynamical model is dissipative and preserves its stability properties even for nonstationary biomedical data.

A notable feature is the linear behavior of V(t) when plotted on a semi-logarithmic scale, which provides clear numerical evidence of exponential decay. This indicates that the error energy relative to the equilibrium state decreases at an approximately constant exponential rate. Furthermore, the time derivative of the Lyapunov function remains non-positive along the trajectory and strictly negative away from the equilibrium, thereby satisfying the Lyapunov stability condition.

Together, these observations confirm that the proposed gradient-flow model exhibits global exponential stability, as predicted by the theoretical analysis.

It is important to emphasize that the stability properties depend solely on the spectral characteristics of the matrix M+W, while the input signal affects only the equilibrium point through the vector d.

Therefore, the convergence behavior remains structurally determined by the system matrix, and not by the particular waveform of the input signal.

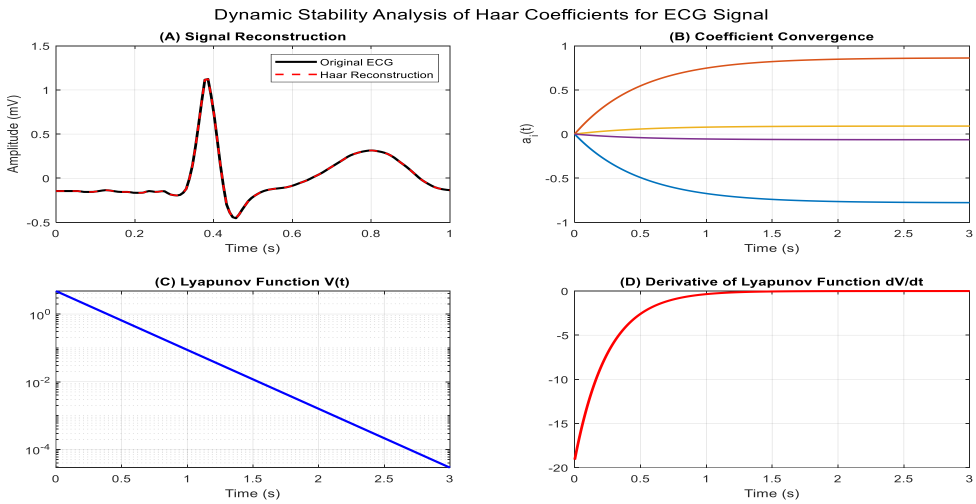

Fig 4:

5. Conclusion and Discussion

In this paper, we reformulated the classical Haar wavelet approximation problem within a continuous-time dynamical systems framework. Starting from the discrete L^2 approximation of a signal using a finite Haar basis, we constructed a quadratic energy functional whose minimizer corresponds to the optimal coefficient vector. Rather than computing this minimizer solely through direct algebraic projection, we modeled the coefficient estimation process as a gradient-flow differential system.

The study proceeded in a structured manner. First, the quadratic representation of the energy functional was derived explicitly, identifying the matrices $M = \Psi^T \Psi, d = \Psi^T f$ and incorporating a symmetric positive semidefinite regularization matrix W. The gradient of the energy was computed rigorously, leading to the linear gradient-flow model $a^{\cdot}(t) = -\gamma(M+W)a(t) + \gamma d$, Second, we established the existence and uniqueness of the equilibrium point $a^* = (M+W)^{-1} d$, under the assumption that M+W is symmetric positive definite. We then derived the closed-form solution of the dynamical system using matrix exponential methods and proved global exponential stability through two independent approaches: spectral analysis and Lyapunov theory. In both cases, an explicit convergence rate was obtained, governed by $\gamma\lambda_{min}(M+W)$. The principal contribution of this work lies not in solving a linear normal equation since the algebraic computation of Haar coefficients is classical and well documented in standard wavelet literature such as A Wavelet Tour of Signal Processing but in introducing a dynamical interpretation of the approximation problem and providing a rigorous stability analysis of the resulting gradient-flow system.

Classical approaches to Haar approximation focus on orthogonal projection formulas or fast wavelet transforms. They typically do not analyze the coefficient computation as a dynamical process, nor do they provide explicit exponential decay estimates or Lyapunov-based stability proofs. The present work differs by embedding the approximation problem into continuous-time stability theory and explicitly linking convergence speed to spectral properties of the system matrix.

The numerical validation considered two representative signals: a synthetic periodic signal and a real electrocardiogram signal consisting of 128 samples obtained from the PhysioNet. In both cases, the orthonormal setting M=I and W=0 was employed. The numerical experiments confirmed the predicted exponential convergence behavior and validated the theoretical decay rate derived analytically.

Although the regularization matrix W was not activated in the numerical examples, its theoretical role was fully characterized. The analysis shows that W≥0 modifies the spectrum of M+W, improves conditioning, and increases the exponential convergence rate. Therefore, the proposed framework naturally extends to regularized and potentially adaptive settings.

In summary, this paper establishes a rigorous bridge between Haar wavelet approximation and continuous-time dynamical systems theory. By interpreting coefficient computation as a gradient-flow process and proving global exponential stability with explicit decay rates, the study extends the classical algebraic viewpoint toward a stability-oriented analytical framework. This dynamical perspective provides structural insight into transient behavior and opens directions for future research involving nontrivial regularization, time-varying models, or nonlinear extensions.

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