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# On the upper hull domination number of a graph

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**Abstract**
For a connected graph $G = (V, E)$, a hull set $S$ in a connected graph $G$ is called a hull dominating set of $G$ if $S$ is both hull set and a dominating set of $G$. The hull domination number $\gamma_h(G)$ of $G$ is the minimum cardinality of a hull dominating set of $G$. A hull dominating set $S$of $G$ is called a *minimal hull dominating set* if there is no proper subset $S$ of S such that $S$ is a hull dominating set of $G$. The *upper hull domination number* $\gamma_h^+(G) = m\,\text{ax}\{|S|: S \text{ is a minimal hull dominating set of } G\}$. Some general properties satisfied by this concept are studied. Connected graphs of order $p \geq 2$ with *upper hull domination number* $p$ *or* $p-1$ are characterized. It is shown that for every positive integer$a \geq 2$, there exists a connected graph $G$ such that $\gamma_h(G) = a$ and $\gamma_h^+(G) = 2a$.

**Keywords:** domination number, hull number, hull domination number, upper hull domination number

## 1. Introduction

Graphs are often used as models for studying structures and relationships in many real-world situations. In general, graphs can be used to model a binary relation system. For a given binary relation system $(V, R)$, the graph $G$ associated with it is that graph with $V(G) = V$ and $E(G) = E = \{uv \,/\, u, v \in V \text{ and } uRv\}$. It is natural to ask how close two objects are to each other under a certain binary relation. The closeness then can be measured by the distance between two vertices in the graph associated with the relation. For a connected graph $G$ and vertices $u$ and $v$ of $G$, we define the *distance* $d(u, v)$ between $u$ and $v$ as the length of a shortest $u - v$ path in $G$. For a set $S$ of vertices, let $I[S] = \cup_{u,v\in S} I[u, v]$. The set $S$ is convex if $I[S] = S$. Clearly if $S = \{v\}$ or $S = V$, then $S$. The *convexity number*, denoted by $C(G)$, is the cardinality of a maximum proper convex subset of $V$. The smallest convex set containing $S$ is denoted by $I_h(S)$ and called the *convex hull* of $S$. Since the intersection of two convex sets is convex, the convex hull is well defined. Note that $S \subseteq I[S] \subseteq I_h(S) \subseteq V$. A The *hull number* $h(G)$ of $G$ is the minimum order of its hull sets and any hull set of order $h(G)$ is a *minimum hull set* or simply a $h$*-set* of $G$. A set of vertices $D$ in a graph $G$ is a *dominating set* if each vertex of $G$ is dominated by some vertex of $D$. The *domination number* of $G$ is the minimum cardinality of a dominating set of $G$ and is denoted by $\gamma(G)$. A dominating set of size $\gamma(G)$ is said to be a $\gamma$-set. A hull set $S$ in a connected graph $G$ is called a *hull dominating set* of $G$ if $S$ is both hull set and a dominating set of $G$. The *hull domination number* $\gamma_h(G)$ of $G$ is the minimum cardinality of a hull dominating set of $G$. Any hull dominating set of $G$ with cardinality $\gamma_h(G)$ is called a $\gamma_h$ *-set* of $G$.These concepts were studied in [1 -26]. For basic definitions and terminologies, we refer to [2, 10]. The order and size of $G$ are denoted by $p$ and $q$ respectively. A vertex is called an *extreme vertex* of $G$ if subgraph induced by its neighbours is complete. We consider connected graphs with at least two vertices. Throughout the following $G$ denotes a connected graph with at least two vertices. The following theorems are used in the sequel.

**Theorem 1.1[1].** For a complete graph $K_p, p \geq 2, \gamma_h(K_p) = p$.
**Theorem 1.2[1].** For the graph $G = K_1 + \cup\, m_jK_j$, where $\sum m_j \geq 2$, $\gamma_h(G) = p - 1$.

## 2. on the upper hull domination number of a graph

**Definition 2.1.** A hull dominating set $S$ of $G$ is called a *minimal hull dominating set* if there is no proper subset $S$ of S such that $S$ is a hull dominating set of $G$. The *upper hull domination number*

$\gamma_h^+(G) = max\{|S|: S \text{ is a minimal hull dominating set of } G\}$

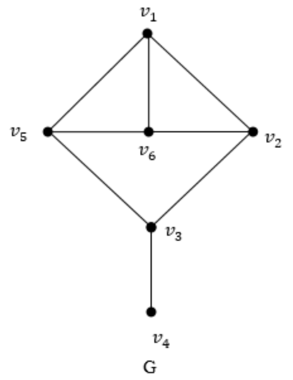


**Fig 2.1**

**Example 2.2.** For the graph $G$ given in Figure 2.1, $S_1 = \{v_1, v_4\}$ is a $\gamma_h$ -set of $G$ so that $\gamma_h(G) = 2$. The set $S_2 = \{v_2, v_4, v_5\}$ is a hull dominating set of $G$. It is easily verified that there is no minimal hull dominating set $S$ of $G$ with $|S| \geq 4$. Hence it follows that $\gamma_h^+(G) = 3$.

**Remark 2.3.** Every minimum hull dominating set of $G$ is a minimal hull dominating set of $G$ and the converse is not true.

**Theorem 2. 4.** For a connected graph $G, 2 \leq \gamma_h(G) \leq \gamma_h^+(G) \leq p$.

**Proof.** Any hull dominating set needs at least two vertices and so$\gamma_h(G) \geq 2$. Since every minimal hull dominating set is a hull dominating set, $\gamma_h(G) \leq \gamma_h^+(G)$. Also, since $V(G)$ is a hull dominating set of $G$, it is clear that $\gamma_h^+(G) \leq p$. Thus $2 \leq \gamma_h(G) \leq \gamma_h^+(G) \leq p$.

**Remark 2.5.** The bounds in Theorem 2.4 are sharp. For the graph $G$given in Figure 2.1,$\gamma_h(G) = 2$. For any non-trivial tree $T, \gamma_h(T) = \gamma_h^+(G)$ and for the complete graph $G = K_p, \gamma_h^+(G) = p$. Also, all the inequalities in Theorem 2.4 are strict. For the graph $G$ given in Figure 2.2, $\gamma_h(G) = 3$ and $\gamma_h^+(G) = 4$. Thus $2 < \gamma_h(G) < \gamma_h^+(G) < p$.

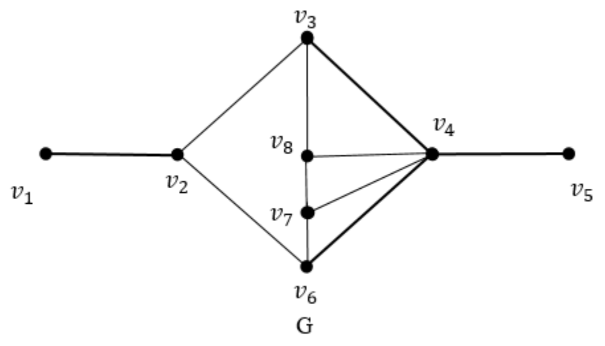


**Fig 2**

**Observation 2.6(i).** Each extreme vertex of $G$ belongs to every hull dominating set of $G$.
**(ii).** For any connected graph$G$, no cut-vertex of $G$ belongs to any minimal hull dominating set of $G$.
**(iii)** For any non-trivial tree $T$, $\gamma_h^+(T) = k$, where $k$ is the number of end vertices of $T$.
**(iv)**. For a complete graph $K_p, p \geq 2, \gamma_h^+(K_p) = p$.

**Theorem 2.7.** For a complete bipartite graph $G = K_{m,n}$ $(m, n \geq 2), S = \{u, v, w\}$ is a minimum hull dominating set of $G$ if and only if $u$ and $v$ are independent and $w$ is adjacent to both $u$ and $v$

**Proof.** Let $S = \{u, v, w\}$ be a minimum hull dominating set of $G$. Suppose $u$ and $v$ are independent and $w$ is adjacent to both $u$ and $v$. Then $I_h[S] = S$ and so $S$ is not a hull dominating set of $G$, which is a contradiction. Conversely, let $S = \{u, v, w\}$ where $u$ and $v$ are independent and $w$ is adjacent to both $u$ and $v$. It is clear that $I_h[S] = V(G)$ so that $S$ is a hull dominating set of $G$. Since $|S| = 3$, $S$ is a minimum hull dominating set of $G$.

**Theorem 2.8.** For a complete bipartite graph$G = K_{m.n}$ $(2 \leq m \leq n)$, $\gamma_h^+(G) = 3$.

**Proof.** Let $U = \{x_{1,}x_{2,}, \dots, x_m\}$ and $W = \{y_{1,}y_{2,}, \dots, y_n\}$ be a bipartition of $G$. Let $S = \{u, v, w\}$ be a minimum hull dominating set of $G$. Suppose $u$ and $v$ are independent and $w$ is adjacent to both $u$ and $v$. Then $S$ is a hull dominating set of $G$, so that $\gamma_h(G) = 3$. We prove that $\gamma_h^+(G) = 3$. If not, let $S_1$ be a minimal hull dominating set of $G$ with $|S_1| \geq 4$. If $S_1$ consists of at least 3 independent vertices of $G$. Then either $S_1$ is a hull dominating set of $G$ or $S_1$ consists of a hull dominating set of $G$, which is a contradiction to $S_1$ a minimal hull dominating set of $G$. Therefore $S_1$ consists of exactly two independent vertices of $G$ say, $u, v$ and $w$ is adjacent to both $u$ and $v$. By Theorem 2.7, $\{u, v, w\}$ is a minimum hull dominating set of $G$, which is a contradiction to $S_1$ a minimal hull dominating set of $G$. Hence $\gamma_h^+(G) = 3$.

**Theorem 2.9.** For a connected graph $G$, $\gamma_h(G) = p$ if and only if $\gamma_h^+(G) = p$.

**Proof.** Let $\gamma_h^+(G) = p$. Then $S = V(G)$ is the unique minimal hull dominating set of $G$. Since no proper subset of $S$ is a hull dominating set, it is clear that $S$ is the unique minimum hull dominating set of $G$ and so $\gamma_h(G) = p$. The converse follows from Theorem 2.4.

**Corollary 2. 10.** For a connected graph $G$ of order $p$, the following are equivalent:
1. $\gamma_h(G) = p$
2. $\gamma_h^+(G) = p$
3. $G = K_p$

**Proof.** This follows from Theorem 1.1 and Observation 2.6(iv).

**Theorem 2.11.** Let $G$ be a non-complete connected graph without cut vertices. Then $\gamma_h^+(G) \leq p - 2$.

**Proof.** Suppose that $\gamma_h^+(G) \geq p - 1$. Then by Corollary 2.10, $\gamma_h^+(G) = p - 1$. Let $v$ be a vertex of $G$ and let $S = V(G)$ `$- \{v\}$ be a minimal hull dominating set of $G$. By Observation 2.4(i), $v$ is not an extreme vertex of $G$. Then there exists $x, y \in N(v)$ such that $xy \notin E(G)$. Since $v$ is not a cut vertex of $G$, $< G - v >$ is connected and also $< G - v >$ contains a geodesic of length at least two. Let $u, u_1, u_2, \dots, u_n, w$ be a geodesic in $< G - v >$ of length at least two. Then $S_1 = S - \{u_1\}$ is a hull dominating set of $G$.

Since $S_1 \subseteq S, S_1$ is not a minimal hull dominating set of $G$, which is a contradiction. Therefore $\gamma_h^+(G) \leq p-2$.

**Theorem 2.12.** For a connected graph $G$, $\gamma_h(G) = p-1$ if and only if $\gamma_h^+(G) = p-1$.

**Proof.** Let $\gamma_h(G) = p-1$. Then it follows from Theorem 2.4, $\gamma_h^+(G) = p$ or $p-1$. If $\gamma_h^+(G) = p$ then by Theorem 2.9, $\gamma_h(G) = p$ which is a contradiction. Hence $\gamma_h^+(G) = p-1$. Conversely, let $\gamma_h^+(G) = p-1$, then it follows from Corollary 2.10 that $G$ is non-complete. Hence by Theorem 2.11, $G$ contains a cut vertex, say $v$. Since $\gamma_h^+(G) = p-1$, it follows from Theorem 2.4 that $S = V(G) - \{v\}$ is the unique minimal hull dominating set of $G$. Therefore $\gamma_h(G) = p-1$.

**Corollary 2.13.** For a connected graph $G$ of order $p$, the following are equivalent:
1. $\gamma_h(G) = p-1$
2. $\gamma_h^+(G) = p-1$
3. $G = K_1 + \cup\, m_j K_j$, where $\sum m_j \geq 2$

**Proof.** This follows from Theorem 1.2 and Theorem 2.12. In view of Theorem 2.4, we have the following realization result.

**Theorem 2.14.** For every positive integer $a \geq 2$, there exists a connected graph $G$ such that $\gamma_h(G) = a$ and $\gamma_h^+(G) = 2a$.

**Proof.** Let $R_i : w_i, x_i, y_i, z_i, w_i$ $(1 \leq i \leq a)$be a copy of a cycle $C_4$. Let $G$ be the graph given in Figure 2.3 is obtained from $R_i$ by adding a new vertex $v$ and the edges $vw_i$, $vy_i$
and $xu_i$ $(1 \leq i \leq a)$. Also join each $u_i$ $(1 \leq i \leq a)$ with each $x_i$ $(1 \leq i \leq a)$. It is easily observed that every minimum hull set contains each $v_i$ $(1 \leq i \leq a)$ and so $\gamma_h(G) \geq a$. Let $W = \{v_1, v_2, \ldots, v_a\}$. Then $I_h[W] = V(G)$ and so $W$ is a hull dominating set of $G$. Hence $\gamma_h(G) = a$. Now, $S = \{w_1, w_2, \ldots, w_a, u_1, u_2, \ldots, u_a\}$ is a hull dominating set of $G$. We show that $S$ is a minimal hull dominating set of $G$. Let $M$ be any proper subset of $S$. Then there exist at least one vertex, say $u \in S$ such that $u \notin M$. First assume that $u = w_i$ for some $i$ $(1 \leq i \leq a)$. Then $I_h[M] \neq V(G)$ and so $M$ is not a hull dominating set of $G$. Next, assume that $u = u_j$ for some $j$ $(1 \leq j \leq a)$. Then also $I_h[M] \neq V(G)$ and so $M$ is not a hull dominating set of $G$. Hence $S$ is a minimal hull dominating set of $G$ so that $\gamma_h^+(G) \geq 2a$. Since every minimum hull dominating set contains exactly each $x_i$ $(1 \leq i \leq a)$. it follows that there is no minimal hull dominating set $X$ of $G$ with $|X| \geq 2a+1$. Thus $\gamma_h^+(G) = 2a$.

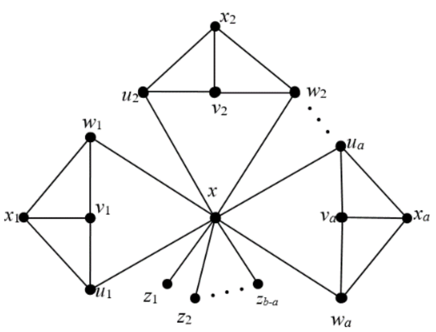


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**Fig 3**

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