International Journal of Multidisciplinary Research and Growth Evaluation

International Journal of Multidisciplinary Research and Growth Evaluation
ISSN: 2582-7138
Received: 01-10-2021; Accepted: 15-10-2021
www.allmultidisciplinaryjournal.com
Volume 2; Issue 6; November-December 2021; Page No. 63-66

On the non-homogeneous ternary cubic equation $3(x^2 + y^2)-2xy+4(x + y)+4 = 51z^3$

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Abstract

The cubic equation with three unknowns given by $3(x^2 + y^2)-2xy+4(x + y)+4 = 51z^3$ is analysed for its different patterns of non-zero distinct integer solutions.

Keywords: Ternary cubic, non-homogeneous cubic, integer solutions

Introduction

The Diophantine equations offer an unlimited field for research due to their variety [1, 2]. In particular, one may refer [3-16] for cubic equations with three unknowns. This communication concerns with yet another interesting equation $3(x^2+y^2)-2xy+4(x+y)+4=51z^3$ representing non- homogeneous cubic equation with three unknowns for determining its infinitely many non-zero integral points.

Method of Analysis

The ternary cubic equation to be solved is

$$3(x^2+y^2)-2xy+4(x+y)+4=51z^3 \tag{1}$$

Introducing the linear transformations

$$x=u+v,\ y=u-v,\quad (u\neq v\neq 0) \tag{2}$$

In (1), it is written as

$$(2u+2)^2+8v^2=51z^3 \tag{3}$$

Now, (3) is solved through different ways and using (2), different sets of integer solutions to (1) are obtained.

Way 1

Assume $z=a^2+8b^2$ (4)

Write 51 as $51=(7+i\sqrt{2})(7-i\sqrt{2})$ (5)

Using (4) and (5) in (3) and applying factorization, it is written as

$((2u+2)+i2\sqrt{2}\,v)((2u+2)-i2\sqrt{2}v))=(7+i\sqrt{2})(7-i\sqrt{2})\,(a+i2\sqrt{2}b)^3\,(a-i2\sqrt{2}b)^3$ Which is equivalent to the system of equations.

$$(2u+2)+i2\sqrt{2}v=(7+i\sqrt{2})(a+i2\sqrt{2}b)^3 \quad (6)$$

$$(2u-2)+i2\sqrt{2}v=(7-i\sqrt{2})(a-i2\sqrt{2}b)^3 \quad (7)$$

Equating the real and imaginary parts in either (6) or (7), the values of *u* and *v* are

$$u=\frac{1}{2}(7a^3-168ab^2-12a^2b+32b^3-2)$$

$$v=\frac{1}{2}(a^3-24ab^2+42a^2b-112b^3)$$

Substituting the above value of u & v in (2), the values of *x* & *y* are given by

$$\begin{aligned} x &= 4a^3-96ab^2+15a^2b-40b^3-1 \\ y &= 3a^3-72ab^2-27a^2b+72b^3-1 \end{aligned} \quad (8)$$

Thus,(8) & (4) represent the non-zero distinct integer solutions to (1).

Note 1

The integer 51 on the R.H.S. of (3) may be expressed as the product of complex conjugates as below:

Choice 1: $51=(-7+i\sqrt{2})(-7-i\sqrt{2})$

Choice 2: $51=(1+i5\sqrt{2})(1-i5\sqrt{2})$

Choice 3: $51=(-1+i5\sqrt{2})(-1-i5\sqrt{2})$

For each of the above choices to 51, the corresponding integer solutions to (1) are given below:

Solutions to choice 1

$$x=-3a^3+72ab^2-27a^2b+72b^3-1$$

$$y=-4a^3+96ab^2+15a^2b-40b^3-1$$

$$z=a^2+8b^2$$

Solutions to choice 2

$$x=3a^3-72ab^2-27a^2b+72b^3-1$$

$$y=-2a^3+48ab^2-33a^2b+88b^3-1$$

$$z=a^2+8b^2$$

Solutions to choice 3

$$x=2a^3-48ab^2-33a^2b+88b^3-1$$

$$y=-3a^3+72ab^2-27a^2b+72b^3-1$$

$$z=a^2+8b^2$$

Way 2

Write (3) as $(2u+2)^2 + 8v^2 = 51z^3 * 1$ (9)

The integer 1 on the R.H.S. of (9) is written as

$$1 = \frac{(-1+i2\sqrt{2})(-1-i2\sqrt{2})}{9} \tag{10}$$

Substituting (5) & (10) in (9) and repeating the process as in Way:1**,** the corresponding integer solutions to (1) are given by

$$x = 9(a^3 - 24ab^2 - 111a^2b + 296b^3) - 1$$
$$y = 27(-4a^3 + 96ab^2 - 15a^2b + 40b^3) - 1$$
$$z = 9(a^2 + 8b^2)$$

Note 2

The integer 1 on the R.H.S. of (9) may be written as follows:

Choice 4: $1 = \frac{(-7+i6\sqrt{2})(-7-i6\sqrt{2})}{121}$

Choice 5: $1 = \frac{(7+i6\sqrt{2})(7-i6\sqrt{2})}{121}$

For each of the above choices to 1, the corresponding integer solutions to (1) are given below:

Solutions to choice 4

$$x = 121(-13a^3 + 312ab^2 - 393a^2b + 1048b^3) - 1$$
$$y = 121(-48a^3 + 1152ab^2 - 27a^2b + 72b^3) - 1$$
$$z = 121(a^2 + 8b^2)$$

Solutions to choice 5

$$x = 5203A^3 - 22143A^2B - 124872AB^2 + 59048B^3 - 1$$
$$y = -726A^3 - 49005A^2B + 17424AB^2 + 130680B^3 - 1$$
$$z = 121A^2 + 968B^2$$

Way 3

Replacing *z* by 2*w* in (3), it is written as

$$(u+1)^2 + 2v^2 = 102\, w^3 \tag{11}$$

Write 102 as $102 = (10+i\sqrt{2})(10-i\sqrt{2})$ (12)

Performing the analysis as given above, the corresponding integer solutions to (1) are found to be

$$x = 11a^3 - 66ab^2 + 24a^2b - 16b^3 - 1$$
$$y = 9a^3 - 54ab^2 - 36a^2b + 24b^3 - 1$$
$$z = 2(a^2 + 2b^2)$$

Note 3

Further, in (11) writing 102 as $102 = (-10 + i\sqrt{2})(-10 - i\sqrt{2})$ and performing a few calculations, the corresponding integer solutions to (1) are given by

$$x = -9a^3 + 54ab^2 - 36a^2 b + 24b^3 - 1$$

$$y = -11a^3 + 66ab^2 + 24a^2 b - 16b^3 - 1$$

$$z = 2(a^2 + 2b^2)$$

Conclusion

To conclude, one may search for other patterns of solutions and their corresponding properties.

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