On the ternary quadratic equation 7 (x^2+y^2) - 13xy = 27z^2

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Abstract
We obtain infinitely many non-zero integer triples $(x, y, z,)$ satisfying the homogenous quadratic equation with three unknowns. Various interesting properties among the values of $x, y,$ and z are presented. Some relations between the solutions and special numbers are exhibited.

Keywords: Ternary quadratic, Homogeneous quadratic, integral solutions, 2010 Mathematics subject classification, 11D09

Introduction
There is a great interest for mathematicians since ambiguity in homogeneous and non-homogeneous Quadratic Diophantine Equations [1-3]. In this context, one may refer [4-15] for varieties of problems on the quadratic Diophantine equations with three, four and five variables. In this paper, quadratic equation with three variables given by $7(x^2+y^2)-13xy=27z^2$ is analyzed for its non-zero distinct integer solutions. A few interesting relations between the solutions and special numbers are exhibited.

Method of analysis
The Diophantine equation representing the ternary quadratic equation to be solved for its non-zero distinct integral solution is

$$7(x^2+y^2)-13xy=27z^2 \tag{1}$$

Substituting the linear transformations

$$x=3u+v, y=3u-v, u\neq v\neq 0 \tag{2}$$

In (1), it leads to

$$u^2+3v^2=3z^2 \tag{3}$$

(3) Can be solved through different methods and we obtain different sets of integer solutions to (1).

Set 1

Assume $z = a^2 + 3b^2$ (4)

Where a and b are non-zero distinct integers.

Write 3 as $3 = (i\sqrt{3})(-i\sqrt{3})$ (5)

Substituting (4) and (5) in (3) and applying the method of factorization, define

$$u + i\sqrt{3}v = (i\sqrt{3})(a + i\sqrt{3}b)^2$$

Equating the real and imaginary parts, we have

$$\left.\begin{aligned} u &= -6ab \\ v &= a^2 - 3b^2 \end{aligned}\right\} \quad (6)$$

From (2), the non-zero distinct integer solutions to (1) are found to be

$$x = -18ab + (a^2 - 3b^2)$$
$$y = -18ab - (a^2 - 3b^2)$$
$$z = a^2 + 3b^2$$

Properties

- $y(2^b, b) + z(2^b, b) + 18(W_b - 1)$ is a nasty number.
- $2\,(2t_{4,a} - x(a, a) - z(a, a))$ *is a perfect square*.
- $x(a,a) + y(a,a) + 36t_{4,a} = 0.$

Note 1

Instead of (5), write 3 as

$$3 = \frac{(3 + i\sqrt{3})(3 - i\sqrt{3})}{4}$$

Following the procedure as in set 1, the corresponding non-zero distinct integer solutions to (1) are obtained as

$$x = 5a^2 - 15b^2 - 6ab$$
$$y = 4a^2 - 12b^2 - 12ab$$
$$z = a^2 + 3b^2$$

Note 2

Instead of (5), write 3 as

$$3 = \frac{(12 + i\sqrt{3})(12 - i\sqrt{3})}{49}$$

By applying the same process as derived in set 1, we have the corresponding non-zero distinct integer solutions to (1) are obtained as

$$x = 259A^2 - 777B^2 + 42AB$$
$$y = 245A^2 - 735B^2 - 294AB$$
$$z = 49\left(A^2 + 3B^2\right)$$

Set 2

One may write (3) as

$$u^2 + 3v^2 = 3z^2 * 1 \tag{7}$$

Write 1 as

$$1 = \frac{(1 + i\sqrt{3})(1 - i\sqrt{3})}{4} \tag{8}$$

Using (4) , (5) and (8) in (7) and applying the method of factorization, define

$$\left(u + i\sqrt{3}\,v\right) = \left(i\sqrt{3}\right)\frac{\left(1 + i\sqrt{3}\right)}{2}\left(a + i\sqrt{3}\,b\right)^2 \tag{9}$$

Equating the real and imaginary parts of (9), we have

$$u = \frac{1}{2}\left\{-3a^2 + 9b^2 - 6ab\right\}$$
$$v = \frac{1}{2}\left\{a^2 - 3b^2 - 6ab\right\}$$

In view of (2), the integer solutions of (1) are given by

$$x = -4a^2 + 12b^2 - 12ab$$
$$y = -5a^2 + 15b^2 - 6ab$$
$$z = a^2 + 3b^2$$

Note 3

- Instead of (11), ‘1’ can also be considered as $1 = \frac{(1 + i4\sqrt{3})(1 - i4\sqrt{3})}{49}$, and the corresponding non-zero solutions are as follows:

$$x = -196A^2 + 588B^2 - 3528AB$$
$$y = -560A^2 + 1680B^2 - 3024AB$$
$$z = 196\left(A^2 + 3B^2\right)$$

- It is worth to mention that, ‘1’ maybe considered in general as

$$1 = \frac{\left\{\left(3r^2 - s^2\right) + i\sqrt{3}\,r\,s\right\}\left\{\left(3r^2 - s^2\right) - i\sqrt{3}\,r\,s\right\}}{\left(3r^2 + s^2\right)^2}$$

Properties

- $3\left\{4z(a, a+1) - x(a, a+1) - 12\,\mathrm{Pr}_a\right\}$ is a nasty number.

- $5z(1,b)+y(1,b)-10t_{4,b}-S_b+1=0.$
- $z(a,a)-x(a,a)+2y(a,a)$ is a perfect square.

Set 3

Equation (3), can also be written as $3z^2-u^2=3v^2$ (10)

Assume $v=3a^2-b^2$ (11)

Write 3 as $3=(2\sqrt{3}+3)(2\sqrt{3}-3)$ (12)

Substituting (11) and (12) in (10) and applying the method of factorization, define

$$\sqrt{3}z+u=(2\sqrt{3}+3)(\sqrt{3}a+b)^2$$

Equating the rational and irrational roots, we get

$$\left.\begin{aligned} u&=9a^2+3b^2+12ab \\ z&=6a^2+2b^2+6ab \end{aligned}\right\} \quad (13)$$

From (2), the non-zero distinct integer solutions to (1) are found to be

$$x=30a^2+8b^2+36ab$$
$$y=24a^2+10b^2+36ab$$
$$z=6a^2+2b^2+6ab$$

Properties

- $x(a+1,a)-5z(a+1,a)+2*t_{4,a}=6*\mathrm{Pr}_a.$
- $6z(2^a+1,1)-y(2^a+1,1)-12Ky_a-26=0.$
- $6\left[x(a,a)-y(a,a)\right]$ is a nasty number.

Note 4

In equation (12), we can choose '3' as $3=(7\sqrt{3}+12)(7\sqrt{3}-12)$.
For this choice, we obtain the non-zero distinct integer solution to (1) are given by

$$x=111a^2+35b^2+126ab$$
$$y=105a^2+37b^2+126ab$$
$$z=21a^2+7b^2+24ab$$

Set 4
Now we are going to introduce another set of transformations as

$$u=X+3T\ ,\ v=X-T\ ,\ z=2W \quad (14)$$

In (3), we obtain

$$X^2+3T^2=3W^2 \quad (15)$$

For this, the solutions are said to be

$$\left.\begin{aligned} W &= a^2 + 3b^2 \\ X &= -6ab \\ T &= a^2 - 3b^2 \end{aligned}\right\} \quad (16)$$

By using (16) in (14), we get

$$u = 3a^2 - 9b^2 - 6ab$$
$$v = -a^2 + 3b^2 - 6ab$$
$$z = 2a^2 + 6b^2$$

In view of (2), the non-zero distinct integer solutions to (1) are found to be

$$x = 8a^2 - 24b^2 - 24ab$$
$$y = 10a^2 - 30b^2 - 12ab$$
$$z = 2a^2 + 6b^2$$

Properties

- $x(a,a) - 2y(a,a)$ is a nasty number.
- $x\left(1,\sqrt{2^b}\right) - 2y\left(1,\sqrt{2^b}\right) - 12Tk_b = 0.$
- $\frac{-1}{8}\left[x(a,a) + 4z(a,a) \right]$ is a perfect square.

Notations Used

- Regular Polygonal Number of rank n with sides m: $t_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right]$
- Pronic Number of rank n : $\Pr_n = n(n+1)$
- Woodall Number of rank n: $W_n = n(2^n) - 1$
- Star Number of rank n : $S_n = 6n(n-1) + 1$
- Thabit-ibn-Kurrah Number of rank n: $TK_n = 3(2^n) - 1$
- Kynea Number of rank n: $Ky_n = (2^n + 1)^2 - 2$

Conclusion

In this paper, we have presented different patterns of integer solutions to the ternary quadratic equation $7\left(x^2 + y^2\right) - 13xy = 27z^2$. As the diophantine equations are rich in variety, one may attempt to find integer solutions to other choices of equations of degree two with three or more unknowns.

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