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Numerical solution of integral equations of convolution

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Abstract
This paper evaluates numerical approximation of Volterra integral equation of convolution type via Laplace transform. The linear and nonlinear both the cases are discussed. The numerical method used is the equal weight quadrature rule. The trapezoidal rule is utilized to approximate inverse Laplace transform using hyperbolic contour. The absolute error between exact and numerical value and error bound are evaluated. Numerical examples illustrate the efficiency and accuracy of present method.

Keywords: Volterra integral equation (VIE), Laplace Transformation, Quadrature method, Trapezoidal rule, Contour, Hyperbolic contour

1. Introduction

The Volterra integral equation plays a vital role in mathematics and engineering. Volterra integral equation is utilized to model different physical phenomena, like population dynamics, continuum mechanics of material with memory, economics problems and spread of epidemics. Also we come across to the Volterra integral equation in the study of fluid dynamics, electrostatics [6], diffusion problems [3] and conceret problem of mechanics.

Various methods have been used to approximate the Volterra integral equation model numerically. For example the expansion methods [12, 14, 18], the interpolation methods [5, 8, 11, 16, 19], the operational matrix methods [1, 2, 20], the iterative methods [4, 7], and the Laguerre transform method [9].

In the present work we used the Laplace transform to solved Volterra integral equation of convolution type. In most of the cases the inverse Laplace transform is hard to evaluate analytically. In such cases we have to approximate the inverse Laplace transform with the help of numerical methods. The trapezoidal rule will be used for approximation inversion of Laplace.

The inversion of Laplace transform itself difficult task to compute. For computing the inversion of Laplace transform efficiently, we need to choose an optimal contour of integration in the complex plane. Various optimal contour of integration to compute efficiently inversion of Laplace transform have been developed in the work of [15, 22]. In present work we used the path due to [15] to construct our numerical scheme for approximating the Volterra integral equation of convolution type. For such types of problems our scheme performed very efficiently and outperformed other numerical schemes for solving Volterra integral equation of convolution type.

2. Preliminaries

Definition 2.1. [21, p.17], A linear Volterra integral equation is defined as

$$\alpha\, v(t) = f(t) + \lambda \int_0^t k(t,s)v(s)ds, \qquad (1)$$

Where α and λ are constants and $k(t,s)$ is some kernel of the integral,

If $\alpha = 0$ and$\lambda = -1$, the (1) become

$$\int_0^t k(t,s)v(s)ds = f(t), \tag{2}$$

If $\alpha = 1$ and$\lambda = 1$, the (1) become

$$v(t) = f(t) + \int_0^t k(t,s)v(s)ds, \tag{3}$$

The equation (2) and equation (3) are known as linear Volterra integral equations of first and second kind respectively.
Definition 2.2. [21, p.238] A Volterra integral equation of the form

$$\alpha\, v(t) = f(t) + \lambda \int_0^t k(t,s)g(v(s))ds, \tag{4}$$

In above equation α and λ are constant is known as nonlinear Volterra integral equation.

For example, $\int_0^t e^{t-s} \ln(u(s))\, ds = e^t - t - 1$, [20] and $e^{2t} - e^t = \int_0^t e^{t-s}u^2(s)ds$ [2].

Definition 2.3. [17, p.499], The Laplace transform of a given function $p(t)$ of a real variable t $(t \geq 0)$ is represented as

$$\mathcal{L}[p(t)] = P(z) = \int_0^\infty e^{-zt} p(t)dt, Re(z) > 0, \tag{5}$$

Where e^{-zt} is the kernel of the transform and $z = s + i\sigma$ is the transformed complex variable.
Definition 2.4. [17, p.450], The inverse Laplace transform of $F(z)$ is defined by

$$f(t) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} F(z)e^{zt}dz, \tag{6}$$

Where $C - i\infty$ to $C + i\infty$ is aline parallel to y-axis and all the possible singularities of $F(z)$ is lies to the right side of it and $> \sigma$.

Definition 2.5. The Convolution of two functions $g_1(t)$ and $g_2(t)$ is represented as

$$g_1(t) * g_2(t) = \int_0^t g_1(t-s)g_2(s)ds\ t \geq 0\,. \tag{7}$$

3. Analysis of the method

Volterra equation of first kind with kernel depending on difference of the arguments has the form [17, p.463]

$$\int_0^t k(t-s)v(s)ds = f(t). \tag{8}$$

Taking the Laplace transform of equation (8) and the convolution theorem we have

$$\mathcal{L}\{k(t)\}.\mathcal{L}\{v(t)\} = \mathcal{L}\{f(t)\}, \tag{9}$$

If we denote $\mathcal{L}\{k(t)\} = K(z)$, $\mathcal{L}\{v(t)\} = V(z)$ and $\mathcal{L}\{f(t)\} = F(z)$, then we have

$$V(z) = \frac{F(z)}{K(z)}\,. \tag{10}$$

The inverse Laplace transform of $V(z)$ gives the solution of $v(t)$ of Volterra equation (8)

$$v(t) = \frac{1}{2\pi i}\int_{C-i\infty}^{C+i\infty} V(z)e^{zt}dz. \tag{11}$$

We have to select the contour to approximate the line $C - i\infty$ to $C + i\infty$, for example parabolic or hyperbolic. The above integrand will be exponentially decayed if we deform $C - i\infty$ to $C + i\infty$ into the left half. The parametric equation of hyperbola is given by [15].

$$z = \omega + \lambda(1 - \sin(\sigma - iu)), -\infty < u < \infty\ (\Gamma) \tag{12}$$

Γ Represents left branch of hyperbola is given below

$$\left(\frac{x-\omega-\lambda}{\lambda\sin\sigma}\right)^2-\left(\frac{y}{\lambda\cos\sigma}\right)^2=1.$$

The equation (11) can be written as

$$v(t)=\frac{1}{2\pi i}\int_{\Gamma}V(z)e^{zt}\,dz\,, \tag{13}$$

Where the Γ represent the contour of integration. Using the above contour $z=z(u)$ given in Γequation (13) becomes

$$v(t)=\frac{1}{2\pi i}\int_{\Gamma}V(z(u))e^{z(u)t}z'(u)du\,, \tag{14}$$

Where $V(z(u))=V(z)$ and using equal weight quadrature rule i.e. trapezoidal rule with $k>0$ here we set

$z_j=z(u_j)$, $z'_j=z'(u_j)$ the equation (14) can be expressed as

$$V_N(t)=\frac{k}{2\pi i}\sum_{j=-N}^{j=N}V(z_j)e^{z_jt}z'_j\,. \tag{15}$$

Similarly we can derive the approximation for the second kind as well as nonlinear Volterra equation of convolution type.

Theorem 3.1. [15] Let v is solution of (8) and $\hat{f}$ is analytic in $\sum_{\beta}^{\omega}$. Let $0<t_0<T, 0<\theta<1$, and let $b>0$ is given as $\cosh b=\frac{1}{\theta\tau\sin\sigma}$ where $=t_0/T$. Let r satisfy $0<r<\min(\sigma,\beta-\pi/2-\sigma)$ so that $\Gamma\subset S_r\subset\sum_{\beta}^{\omega}$, and let the scaling factor be $\lambda=\theta\tilde{r}N/bT$. Therefore, we have for the approximate solution $V_N(t)$ defined by (15) with $k=b/N\le\tilde{r}/\log 2.$ $\|v_N(t)-v(t)\|\le CMe^{\omega t}l(\rho_rN)e^{-\mu N}(\|v_0\|+\|\hat{f}\|\sum_{\beta}^{\omega})$, for $t_0<t<T$, where $\mu=\tilde{r}(1-\theta)/b$, $\rho_r=\theta\tilde{r}\tau\sin(\sigma-r)/b$, $=C_{\sigma,r,\beta}$, $\tilde{r}=2\pi r$ and $l(\rho_rN)=\max(1,\log(1/\rho_r N))$.

We will use the above theorem to find the error bound corresponding to the contour Γ in our numerical experiments.

4. Application of proposed numerical scheme

In this section we validated our numerical scheme by solving various types of Volterra integral equation linear as well as nonlinear of convolution type. All the numerical result obtained using the following values of optimal parameters, $t=0.1, T=1, t_0=0.01, \theta=0.1, \sigma=0.3812, \tau=t_0/T, b=\cosh^{-1}(1/\theta\tau\sin(\sigma)), r=0.3431, \tilde{r}=2\pi r, k=b/N, \omega=02, \lambda=\theta\tilde{r}N/bT$. In Examples 1-5 we solved linear Volterra integral equations and in Examples 6-7, we solved the nonlinear Volterra integral equations. Let $E_N(t)$ represents absolute error between numerical and exact value using the contours Γ and $l(\rho_rN)e^{-\mu N}$ represent the error bound for contour Γ.

4.1 Example 1.

We consider the following integral equation [1].

$$\int_0^t\cos(t-s)u(s)ds=\sin(t), \tag{16}$$

We approximate this problem using the method discussed in section 3. The exact solution of this problem is $u(t)=1$, the error between numerical and exact solution are given in Table 1.

Table 1: Approximate solution using the present method in terms of actual error and estimated error corresponding to Example 1.

N	$E_N(t)$	$l(\rho_rN)e^{-\mu N}$
20	7.3566e-004	9.3500e-002
30	3.5890e-005	9.3000e-003
32	2.1109e-005	5.9000e-003

40	2.0712e-005	9.3775e-004
50	1.1044e-007	9.5200e-005
60	6.7561e-009	9.7094e-006
70	3.6950e-010	9.9351e-007
80	2.3408e-011	1.0191e-007
90	1.2895e-012	1.0473e-008
100	8.3455e-014	1.0779e-009
110	4.9641e-015	1.1107e-010
120	4.6967e-016	1.1458e-011
128	1.1509e-016	1.8629e-012
[18] 6.6000e-014		
Method 1 [19] 2.4000e-014		
Method 2 [19] 8.1000e-013		

4.2 Example 2.
Here we consider our second example [1, 13] to validate our numerical scheme

$$\int_0^t \cos(t-s)u(s)ds = \text{tsin}(t), \tag{17}$$

The exact solution of Example 2 is $u(t) = 2\sin(t)$ the actual errors and estimated errors are given in Table 2.

Table 2: Approximate solution using the present method in terms of actual error and estimated error corresponding to Example 2.

N	$E_N(t)$	$l(\rho_r N)e^{-\mu N}$
16	1.6225e-004	2.3670e-001
32	1.6305e-007	5.9000e-003
48	1.0698e-009	1.5036e-004
64	8.4013e-012	3.8998e-006
80	7.1641e-014	1.0191e-007
83	2.7800e-015	5.1485e-008
88	6.8996e-015	1.6506e-008
93	3.5561e-015	5.2936e-009
96	1.1657e-015	2.6760e-009
112	8.8819e-016	7.0515e-011
122	1.9792e-016	7.2754e-012
125	3.0531e-016	3.6813e-012
126	7.4949e-016	2.9335e-012
128	7.3501e-017	1.8629e-012
[1] 2.0970e-003		
[10] 1.5000e-008		
[13] 2.7700e-015		
[18] 1.9000e-013		
Method 1 [19] 6.5000e-011		
Method 2 [19] 5.4000e-013		

4.3 Example 3.
In this example we take the following integral equation [2].

$$\int_0^t e^{t-s} u(s)ds = t, \tag{18}$$

The exact solution of the problem is $u(t) = 1 - t$, and the results are given in the Table 3.

Table 3: Approximate solution using the present method in terms of actual error and estimated error corresponding to Example 3.

N	$E_N(t)$	$l(\rho_r N)e^{-\mu N}$
32	2.1990e-005	5.9000e-003
48	2-0808e-007	1.5036e-004
64	2.1723e-009	3.8998e-006
80	2.3444e-011	1.0191e-007
96	2.5848e-013	2.6760e-009
112	3.9630e-015	7.0515e-011
128	4.4593e-016	1.8629e-012
144	3.5530e-015	4.9312e-014
[2] 3.1300e-002		

4.4 Example 4. [2]

$$u(t) = t + \frac{4}{3}t^{\frac{3}{2}} - \int_0^t \frac{u(s)}{\sqrt{t-s}} ds, \tag{19}$$

Taking Laplace of equation (19) and using convolution theorem we have

$$\mathcal{L}\{u(t)\} = \mathcal{L}\{t\} + \mathcal{L}\left\{\frac{4}{3}t^{\frac{3}{2}}\right\} - \mathcal{L}\left\{\frac{1}{\sqrt{t}} * u(t)\right\}, \tag{20}$$

$$\mathcal{L}\{u(t)\} = \mathcal{L}\{t\} + \mathcal{L}\left\{\frac{4}{3}t^{\frac{3}{2}}\right\} - \mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\}.\mathcal{L}\{u(t)\}, \tag{21}$$

And we know that

$$\mathcal{L}\{t\} = \frac{1}{z^2}, \mathcal{L}\{u(t)\} = u(z), \left\{t^{\frac{3}{2}}\right\} = \frac{3\sqrt{\pi}}{4\, z^{\frac{5}{2}}} \text{ and } \left\{t^{\frac{-1}{2}}\right\} = \frac{\sqrt{\pi}}{z^{\frac{1}{2}}}. \tag{22}$$

Using the above result we have

$$u\{z\} = \frac{1}{z^2}. \tag{23}$$

The exact solution is $u\{t\} = t$, the error between numerical and exact value are given in the Table 4 using equations (15), (Γ) and also error bound for (Γ) at $t = \frac{1}{8}$.

Table 4: Example 4

N	$E_N(t)$	$l(\rho_r N)e^{-\mu N}$
16	4.0041e-005	2.3670e-001
32	1.6316e-007	5.9000e-003
48	8.1636e-010	1.5036e-004
64	2.5757e-012	3.8998e-006
80	1.6734e-014	1.0191e-007
96	3.3244e-016	2.6760e-009
112	6.6617e-016	7.0515e-011
128	2.8057e-017	1.8629e-012
144	7.9149e-016	4.9312e-014
160	1.1936e-015	1.3074e-015
[2] 3.1200e-002		

4.5 Example 5. [1, 10, 13]

$$\int_0^t e^{t+s} u(s) ds = te^t, \tag{24}$$

Using Leibnitz generalized formula [21, p.30], the equation (24) become

$$u(t) + \int_0^t e^{s-t} u(s) ds = e^{-t}(1+t), \tag{25}$$

The equation (25) also be written as $u(t) + \int_0^t e^{-(t-s)} u(s) ds = e^{-t}(1+t)$. (26)

Taking Laplace of equation (26) we have

$$\mathcal{L}\{u(t)\} + \mathcal{L}\{e^{-t} * u(t)\} = \mathcal{L}\{e^{-t}(1+t)\}, \tag{27}$$
$$\mathcal{L}\{u(t)\} + \mathcal{L}\{e^{-t}\}.\mathcal{L}\{u(t)\} = \mathcal{L}\{e^{-t}\} + \mathcal{L}\{te^{-t}\}, \tag{28}$$

And we know that

$$\mathcal{L}\{e^{-t}\} = \frac{1}{z+1}, \mathcal{L}\{te^{-t}\} = \frac{1}{(z+1)^2} \text{ and } \mathcal{L}\{u(t)\} = u(z). \tag{29}$$

Using the above result we have

$$u(z) = \frac{1}{z+2} + \frac{1}{(z+1)(z+2)}, \tag{30}$$

or

$$u(z) = \frac{1}{z+1}. \tag{31}$$

The exact solution is $u\{t\} = e^{-t}$, the error between numerical and exact value are given in the Table 4 using equations (15), (Γ) and also error bound for (Γ).

Table 5: Example 5

N	$E_N(t)$	$l(\rho_r N)e^{-\mu N}$
16	2.5097e-003	2.3670e-001
32	2.1189e-005	5.9000e-003
48	2.0808e-007	1.5036e-004
53	2.2050e-009	4.7976e-005
64	2.1723e-009	3.8998e-006
80	2.3444e-011	1.0191e-007
96	2.5770e-013	2.6760e-009
112	4.6315e-015	7.0515e-011
128	5.5512e-016	1.8629e-012
144	3.5531e-015	4.9312e-014
160	3.8859e-015	1.3074e-015
[1] 9.3100e-004		
[10] 4.7000e-006		
[13] 1.1100e-014		

4.6 Example 6. [2]

Consider the nonlinear VIE

$$e^{2t} - e^t = \int_0^t e^{t-s} u^2(s)ds, \tag{32}$$

Let$u^2(s) = v(s)$, the equation (32) become

$$e^{2t} - e^t = \int_0^t e^{t-s} v(s)ds, \tag{33}$$

Taking Laplace of equation (33) we have

$$\mathcal{L}\{e^{2t} - e^t\} = \mathcal{L}\{e^t * v(t)\}, \tag{34}$$

$$\mathcal{L}\{e^{2t}\} - \mathcal{L}\{e^t\} = \mathcal{L}\{e^t\}.\mathcal{L}\{v(t)\}. \tag{35}$$

We know that

$$\mathcal{L}\{e^{2t}\} = \frac{1}{z-2}, \mathcal{L}\{e^t\} = \frac{1}{z-1} \text{ and } \mathcal{L}\{v(t)\} = v(z), \tag{36}$$

Using the above result, equation (35) become

$$v(z) = \frac{1}{z-2}, \tag{37}$$

$$U_N(t) = \sqrt{V_N(t)}, \tag{38}$$

Where $V_N(t)$ is given in equation (15) and exact solution is $u(t) = e^t$. The error between numerical and exact value are given in the Table 6 using equation (38), (Γ) and also error bound for (Γ).

Table 6: Example 6

N	$E_N(t)$	$l(\rho_r N)e^{-\mu N}$
16	1.1000e-003	2.3670e-001
32	9.4769e-006	5.9000e-003
48	9.3412e-008	1.5036e-004
64	9.7706e-010	3.8998e-006
80	1.0558e-011	1.0191e-007
96	1.2388e-013	2.6760e-009
112	3.9176e-015	7.0515e-011

128	2.2388e-016	1.8629e-012
144	2.8866e-015	4.9312e-014
160	4.9406e-015	1.3074e-015
[2] 3.2800e-002		

4.7 Example 7. [20]

Consider the nonlinear VIE

$$\int_0^t e^{t-s} \, ln\big(u(s)\big) \, ds = e^t - t - 1, \quad (39)$$

let$\ln\big(u(s)\big) = v(s)$, the equation (39) become

$$\int_0^t e^{t-s} \, v(s)ds = e^t - t - 1. \quad (40)$$

Taking Laplace of equation (40) we have

$$\mathcal{L}\{e^t * v(t)\} = \mathcal{L}\{e^t - t - 1\}, \quad (41)$$

$$\mathcal{L}\{e^t\}.\mathcal{L}\{v(t)\} = \mathcal{L}\{e^t\} - \mathcal{L}\{t\} - \mathcal{L}\{1\}, \quad (42)$$

And we have

$$\mathcal{L}\{e^t\} = \frac{1}{z-1}, \mathcal{L}\{t\} = \frac{1}{z^2}, \mathcal{L}\{1\} = \frac{1}{z} \text{ and } \mathcal{L}\{v(t)\} = v(z). \quad (43)$$

Using the above result equation (42) become

$$v(z) = \frac{1}{z^2}, \quad (44)$$

$$U_N(t) = e^{V_N(t)}, \quad (45)$$

Where $U_N(t)$ is given in equation (15). The exact solution is $u\{t\} = e^t$, the error between numerical and exact value are given in the Table 7 using equations (15), (Γ) and also error bound for (Γ).

Table 7: Example 7

N	$E_N(t)$	$l(\rho_r N)e^{-\mu N}$
16	1.4818e-005	2.3670e-001
32	8.8652e-008	5.9000e-003
48	5.9114e-010	1.5036e-004
64	4.6424e-012	3.8998e-006
80	3.9724e-014	1.0191e-007
96	6.9564e-016	2.6760e-009
112	6.6620e-016	7.0515e-011
128	3.0019e-017	1.8629e-012
144	8.8841e-016	4.9312e-014
160	1.1103e-015	1.3074e-015

5. Conclusion

In this paper we proposed a Laplace transform based numerical method coupled with quadrature rule with high accuracy. The method has more accuracy than the Lagrange interpolation, integral expansion, operational matrix with block-pulse function and piecewise constant orthogonal function. The present method is also having better accuracy then optimal homotopy asymptotic method and numerical solution by using recursive scheme. The method is applicable to Volterra integral of convolution type.

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