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The analysis solution of Non-liner Dirac equation

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Abstract

We construct nonlinear extensions of Dirac relativistic electron equation that preserve its other desirable properties such as locality reparability conservation of probability and Poincare invariance we determine the constraint that the nonlinear term must obey and classify the resultant non polynomial nonlinearities in a double expansion in the degree of nonlinearity and number of derivatives. The aims of this paper is to solve the nonlinear Dirac equation using the analysis method. We followed the historical analysis mathematical method and we found the following some results: Possibility of solving the nonlinear Dirac equation for many –bodies system with may bring about more insight into physical system.

Keywords: analysis solution, nonlinear Dirac equation

1. Introduction

The basic of relativistic quantum mechanics was formulated by Paul Dirac in 1928 in away consistent with special relativity [16].The equation describes the behavior of weakaly coupled electrons at high speeds or strongly coupled electrons such as in the case of core electron states in heavy atoms. Among the benefits of this relativistic formulation is the natural emergence of the electron spin and prediction partner to the electron the positron which was discovered experimentally few years later. The physics and mathematics of the Dirac equation is very rich illuminating and provides atheoretical framework for different physical phenomena that are not present in the nonrelativistic regime such as the klein paradox [22]. In addition Dirac equation emerges in the study of the transport properties ingraghene which makes it important for future applications. Exact solution of the Dirac equation with agiven potential configure ationare limited and not trivial compared to the nonrelativisitic Schrodinger equation in fact the Dirac Hamilton being amatrix the spinor space allows for more structure in the potential interaction the terminology given to relativistic problem such as the " Dirac –coulomb" "Dirac-oscillation" "Dirac-Morse" "…."etc refers to the Dirac equation that reduces to an effective Schrodinger –like equation with the nomad potential for the large spinar component different approaches were developed to generate exact solution to the Dirac equation such as superymmetric quantum mechanics and factorization method to only few[29].

2. Analysis

Lemma (2.1)

Let $f^{/}(x^1 \dots\dots\dots\dots\dots x^n), \dots\dots\dots f^n(x^1 \dots\dots\dots\dots\dots\dots\dots x^n)$be c^{∞}function $n$variabels $x^2 \dots\dots\dots.., x^n$defind in aneighorhood of $x_{\circ}^1 \dots\dots.. x_{\circ}^nletd(x^1 \dots\dots.. x^n)=$ det $\left(\frac{df^i}{dx^j}\right)$Bethe Jacobian determinant and suppose that $J(x_{\circ}^1 \dots\dots\dots. x_{\circ}^n) \neq 0$ let $w_0^1 f^1 x_{\circ}^1 \dots\dots.. x_{\circ}^n$,…….$w_{\circ}^n = f_n(x_{\circ}^1 \dots\dots x_{\circ}^n)$. then there is neighborhood $v$of $(x_{\circ}^n, \dots\dots.., x_{\circ}^n)$and there are$nc^{\infty}$functions $(w^1, \dots\dots.., w^n)$, $g^n(w^1, \dots\dots., w^n)$satisfying$g^1(w_{\circ}^1, \dots\dots, w_{\circ}^n)$such that for any $(w^1, \dots., w^n)$.

In this neighborhood the equation$(w)^1 = f_1(x^1, \dots.., x^n),,,,,,, w^n =, f_n(x^1, \dots.., x^n)$

Havethe unique solution $x^1 = g^1(w^1, \dots, w^n) \dots x.,^n = g^n(w^1, \dots \dots, w^n)$ suchthat $(x^1, \dots, x^n) \in$ v [21].

i. Phase Plane Analysis

Example (2.1)

(Pendulum equation for small oscillations) this is the well know equation $x^{\cdot\cdot} + kx = 0, k > 0$.Solving the equation to obtain the solution explicitly we see that every solution is periodic. The same conclusion may by reached analyzing which turns out to be $(x^{\cdot}(t))^2 = k(x(t))^2 = 2E$And E is obtained from the initial conditions the orbits in this case are ellipses (circles if K=1) surrounding the origin in the phase plane [1].

3. Palan of Qualitative Analysis

Plan (3.1)

Let us now formulate a plan to qualitatively study a system of two differential equations with two variables and consider several example we study the system:

$$\begin{cases} \frac{dx}{dt} = f(x, y) \\ \frac{dy}{dt} = g(x, y) \end{cases} \quad (1)$$

Our main aim is to plot the phase portrait of this system and then predict it dynamics. Based on methods which we have developed we will do it two steps

1. Null- Cline Analysis
2. Jacobian Analysis
3. Null- Cline Analysis

We assume that on the OXY- plane the X –axis is the horizontal axis and the Y- axis in the vertical axis.

1. Draw the $\frac{dx}{dt} = 0$ null- clines form the equation $f(x, y) = 0$ using dashed line and the $\frac{dy}{dt} = 0$ nuall –clines form the equation $g(x, y = 0)$ using sold line.
2. Chooseapaint in one of the regions on the X, Y plane and find the vector field for the X-Component denote the X component as dashed→ $if(x, y) > 0$and as adashed ← $if(x, y) < 0$.
3. Find the vector filed for the Y-component at the same point.
   Denote the Y –component as asoiled↑if $g(x, y) > 0$and as asoild ↓ $if g(x, y) < 0$.
4. Fined the vector field in the adjacent regions using the following rule

A. Change the direction of the dashed component of the vector field if to get to the adjacent region you cress the dashed null-cline.

B. Change the direction of the soleid component of the vector field if to get to the adjacent region you cross the solid null-cline.

C. Show the direction of the vector field on the null-clines close to the equilibrium [2].

4. The Nonlinear Dirac Equation

To meet the need of simulating self- ineracting Dirac fermions [14, 15, 27]. The nonlinear Dirac equation (NLDE) was introduced in 1938 [20] which has the form [11].

$$ih\partial_t\varphi = \left[-ich\sum_{j=1}^{3}\alpha_j d_j + mc^2\right]\varphi + e\left[v(x, y)I_4 - \sum_{j=1}^{3} A_j(t, x)\alpha_j\right]\varphi + f(\varphi)\varphi x \in R^3 \quad (2)$$

The nonlinear Dirac equation (3.1) is similar to the Dirac equation except for the nonlinear term (φ). The nonlinearity is introduced for self-interction and in the resulting field equation it is cubic with respect to the wave function which is only signifiacant at extremely highdensities at extremely highdensities. There have been different cubic nonlinearities generated from different application {17}.Here we take $f(\varphi) = g_1(\varphi^*\beta\varphi)\beta + g_2|\varphi|^2 I_4 g_1 g_2 \in R$Tow constants and$\varphi^* = \varphi^{-t}$ while f^- denotes the complex conjugate of f The first term, i.e $g_1 = 0\ and g_2 \neq 0$ is generated from and second term i.e $g_1 = 0\ and g_2 \neq 0$ is generated fom BECS with chiral confinement and/or spin –obet coupling [10, 19]. Aremark is given here that our numerical methods and their error estimates in thesis can be easily extended to the NLDE with other nonlinearities [26, 27] in fact the fact the NLDE has also been in the Einstein –cartan –sciama- kibble theory of gravity in order to extend general relativity to matter with intrinsic angular momentum (spin). And recently, the NLDE has been adapted as amean field model for Bose –Einstein condenstes (BECS) and /or cosmology [25]. Moreover the experimental advances in BECS graphene and other 2D materials have also stimulaterd the research interests on the mathematical analysis and numerical simulations of the Dirac equation and/or the NLDE without /with electromagnetic potentials especially the honeycomb lattice potential [1, 12, 13].

Similar to the process of Dirac equation [5] through apropernondimensionalization with the choice of $x_s, t_s = \frac{mx^2}{h}, As = \frac{mv^2}{e} and\varphi_s = xs^{\frac{-3}{2}}$ as the dimensionless lengh unit time unit potential unit and spinor filed unit respectively and dimension reduction we cane obtain the dimensionless NLDE in- dimensions (d=3,2,1) $id_1\varphi = \left[-\frac{i}{\epsilon}\sum_{j=1}^{d}\alpha_j\partial_j + \frac{1}{\epsilon^2}\beta\right]\varphi + \left[v(t, x)I_4\sum_{j=1}^{d} A_j(t, x)\alpha_j\right]\varphi + f(\varphi)\varphi x \in R^d$ (3)

Whereϵisadimensioniess parameter parameter inversely proportional to the light speed given by

$$0 < \epsilon = \frac{x_s}{t_{sc}} = \frac{v}{c} \leq 1 \tag{4}$$

With $v = \frac{x_s}{t_s}$ the wave speed and

$$f(\varphi) = \lambda_1(\varphi^*\beta\varphi)\beta + \lambda_2|\varphi|^2 I_4\varphi \in c^4 \tag{5}$$

Where $\lambda_1 = \frac{g_1}{mv^2x_s^3} \in Rand\lambda_2 \frac{g_2}{mv^2x_s^3} \in R$ are two dimensioniess constats for the for the interaction to study the dynamics we give the initial condition $\varphi(t = 0, x) = \varphi_\circ x \in R^d$the NLDE (3) is dispersive and time symmetric [29] similar to the Dirac equation after introducing proper total probability density p as well as the current density $J(X, Y) = (J_1(t, x)), J_2(t, x)$ we could the conservation law moreover the NLDE (3.2)Consrves the total mass. thecnergy is conserved if the electromagnetic potentials are time. independenti.e if v(t,x) =v(x) and $A_j(t, x) = A_j(x) forj = 1,2,3$ then E(t)= $\int_{R^d}\left[-\frac{i}{\epsilon}\sum_{j=1}^{d}\varphi^*\alpha_j\partial_j\varphi + \frac{1}{\epsilon^2}\beta\varphi + v(x)|\varphi|^2 + G(\varphi) - \sum_{j=1}^{d}A_j(x)\varphi^*\alpha_j\varphi\right]dx$ (6)

Where

$$G(\varphi) = \frac{\lambda_1}{2}(\varphi^*\beta\varphi)^2 + \frac{\lambda^2}{2}|\varphi|^4\varphi \in c^4 \tag{7}$$

In (3,2) if the external electromagnetic potential potential are taken to be constants i.e v(t,x)≡ $v^\circ$and $A_j(t, x) \equiv A_j^\circ forj = 1,2,3$ then the NLDE (2) admits the plane wave solution as $\varphi(t, x) = Be^{i(k-x-wt)}$where the time frequency W amplitude vector $B \in R^4$ and spatial wave number $k = (k_1, \dots, k_d) \in R^d$ satisfy.

$$w = \left[\sum_{j=1}^{d}\left(\frac{k_j}{\epsilon} - A_j^\circ\right)\alpha_j + \frac{1}{\epsilon^2}\beta + v^\circ I_4 + \lambda_1(B^*\beta B)\beta + \lambda_2|B|^2 I_4\right]B \tag{8}$$

Which immediately gives the dispersion relation of the dispersion relation of the NLDE (2) as

$$w = w(k, B) = V^\circ + \lambda_2|B|^2 \pm \frac{1}{\epsilon^2}\sqrt{[1 + \epsilon^2\lambda_1(B^*\beta B)] + \epsilon^2|k - \epsilon A^\circ|^2}k \in R^d \tag{9}$$

Again similar to the Dirac equation [5] for one dimension (1D) and two dimension (2D) the NLDE (3.2) can be simplified to the following one

$$id_t\Phi = \left[-\frac{i}{\epsilon}\sum_{j=1}^{d}\sigma_j d_j + \frac{1}{\epsilon^2}\sigma_3\right]\Phi + \left[v(t, x)I_2 - \sum_{j=1}^{d}A_j(t, x)\sigma_j\right]\Phi + f(\Phi)\Phi x \in R^d \tag{10}$$

Where

$$f(\Phi) = \lambda_1(\Phi^*\sigma_3\Phi)\sigma_3 + \lambda_2|\Phi|^2 I_2\Phi \in c^2 \tag{11}$$

With $\lambda_1$and $\lambda_2$both real numbers in (3.9) the two –component wave functions Φdefined as $\Phi = \Phi(t, x) = (\Phi_1(t, x)), \Phi_2(t, x)^t \in c^2$the initial condtion for dynamics is given as [19].

$$\Phi(t = 0, x) = \Phi_\circ(x)x \in R^d \tag{12}$$

The NLDE (8) has similar properties to its four – component ersion (2) it is dispersive and time symmetric satisfies the conservation law{7} conserves total mass and also conserves energy

$$E(t) = \int_{R^d}\left(-\frac{i}{\epsilon}\sum_{j=1}^{d}\Phi^*\sigma_j d_j\Phi + \frac{1}{\epsilon^2}\Phi^*\sigma_3\Phi + v(x)|x|^2 - \sum_{j=1}^{d}A_j(x)\Phi^*\sigma_j\Phi + G(\Phi)\right)dx \tag{13}$$

Where

$$G(\Phi) = \frac{\lambda_1}{2}(x^*\sigma_3\Phi) + \frac{\lambda_2}{2}|\Phi|^4\Phi \in c^2 \tag{14}$$

If the electromagnetic potenials time –independent.

Under constant external electromagnetic potentials i.e$v(t, x) \equiv v^\circ andA_j(t, x) \equiv A_j^\circ forj = 1,2$ the NLDE (8) admits the plane we soluation as $\Phi(t, x) = Be^{i(k.x-wt)}$ with the time frequency w amplitude vector $B \in R^2$ and spatial wave number $k = (k_1, , , , , k_d)^t \in R^d$ satify.

$$wB = \left[\sum_{j=1}^{d}(\frac{ki}{\varepsilon} - A_j^{\circ})\,\sigma_j + \frac{1}{\varepsilon^2}\sigma_3 + v^{\circ} I_2 + \lambda_1(B^*\sigma_3 B)\sigma_3 + \lambda_2|B|^2 I_2\right] B \quad (15)$$

With implies dispersion relation of the NLDE (3.9) directilyas

$$w = w(k, B) = v^{\circ} + \lambda_2|B|^2 \pm \frac{1}{\varepsilon^2}\sqrt{(1 + \varepsilon^2\lambda_1(B^*\sigma_3 B))^2 + \varepsilon^2|k - \varepsilon A^{\circ}|^2} k \in R^d \quad (16)$$

The NLDE (2) has different regimes with different choices of the dimensionless parameter ε. When $\varepsilon = 1$ wich corresponds to the classical regime extensive analytical and numerical result have been obtaind [3, 4, 9].

5. Solution of Nanlinear Dirac Equation

Theorem (5.1)

Let (W),(v) and $(f_\circ) - f_2, (f_3^/), f_4^/$ be satisfied. then (D) v has at lest one nontrivial $u \in \bigcap_{t\geq 2} w^{i,t}(R^3, C^4)$ if in addition f is even and (f_5) hold then $(D)_v$ has infinitely may geometrically distince soluation $U \in \bigcap_{t\geq 2} w^{I,T}(R^4, C^4)$.Here are example where the assumption apply.

Example (5.2)

a. $f(x, u) = \frac{1}{2}b(x)|u|^2\left(1 - \frac{1}{\ln(e+|u|)}\right)$.
b. $f(x, u) = b(x)\psi\left(\frac{1}{2}\right)|u|^2$ where $\psi = [0, \infty) \to [0, \infty)$ is of classc^2 with $\varphi(0) = \varphi^/(0) = 0\, and \varphi^/(s) \to 1$ as $s \to \infty \varphi^{//}(s) \geq 0$
c. $f_u(x, u) = f(x, |u|)u$ where $f(x, s)$ is even in s $f(x, s) \to 0\, ass \to 0$ uniformly in x,f(x,s) is nondecreasing for $s \in [0, \infty) and f(x, s) \to b(x) ass \to \infty$.

6. Variational Setting

We denoted by $|\cdot|_p$ the usual L^p norm for $p \in [1, \infty]$. For $v \in l_{100}^2(R^3, R) the$ operator $A = -i\sum_{K=1}^{3} a_k\partial_k + (v(x) + a)\beta$ is selfadajoint operator in $l^2 = l^2(R^3, C^4)$ it is unbounded from above and from below in orden to inverstigate the spectrum of A we consider

$$A^2 = \Delta + (V + a)^2 + i\sum_{k=1}^{3}\beta a_k d_k v$$

Let $\sigma(s), \sigma_d(s), \sigma_e(s) and \sigma_c(s)$ denote respectively the spectrum the essential spectrum and the continuous spectrum of aself – odjointoperator S on L^2.

Result

We gave several examples of such equation different in structure from those studied previously in the literature and discussed their properties we also demonstrated that our equation were not guge equivalent to the linear Dirac equation.Solution to our nonlinear equation similar to what has been done for simpler polynomial type nonlinear Dirac equation.

Conclusion

We found possibility of solving the nonlinear Dirac equation for many –bodies system using analysis solution.

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