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Extensions of commutative rings

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Abstract
We discovered numerous intriguing types of ring extensions by examining the lowest prime spectra of commutative rings with identity. We investigate what kinds of ring extensions will result in hull-kernel and inverse topologies on the minimal prime spectra as homeomorphisms. These sorts of extensions are compared to other forms of extensions.

Keywords: Rigid extensions, r-extensions, r*-extensions, minimal prime space, Annihilator condition

1. Introduction

We make the implicit premise that all rings are commutative, reduced, and have an identity throughout this essay. R and S will be used to denote such rings for the most part. I 'R is a notation that denotes that I is an ideal of R. L will stand for the aggregation of all R ideas (R). For a subset I ⊆ R we denote the annihilator of I by AnnR (I) = {a ∈ R: ar = 0 for all r ∈ I}. When I = {a1,...,an} we instead write AnnR (a1,...,an). When R is a subring of S (and possessing the identity) we shall write R → S and call this an extension of rings.

Spec denotes the collection of all R prime ideals (R). The typical Zorn's Lemma argument guarantees the existence of minimal prime ideals, which we refer to as Min (R). We will consider Spec as a functor from the category of commutative rings with identity to the category of topological spaces on occasion. Remember that Spec(R) has the hull-kernel topology (a.k.a. the Zariski topology). An arbitrary open set of Spec(R) is one of the form U(I) = P ∈ Spec(R) I P for some I R. When I = {a}, we instead write U(a). The collection {U(a)}a∈R is a base for the hull-kernel topology on Spec(R). We're interested in looking at rings through the lens of their minimal prime ideal space. [7], a paper to which we owe a tremendous debt, was one of the first studies to kick off this programme. As a result, we'd like to dedicate this work to Melvin Henriksen's memory. Huckaba's book [8] also provides a good coverage of the subject, and we use it as our main source. The following lemmas will be applied implicitly.

Lemma 1.1. (Corollary 2.2, [8]) Let R be a reduced ring and let P ∈ Spec(R). P ∈ Min(R) if and only if for each x ∈ P there exists an r ∈ R \ P such that xr = 0.

Lemma 1.2. (Corollary 2.3, [8]) Let R be a reduced ring and let P ∈ Min(R). For a finitely generated ideal I of R, I ⊆ P if and only if AnnR (I) P.

As Min(R) ⊆ Spec(R) it follows that the collection of sets of the form U(a) ∩ Min(R) is a base for the subspace topology on Min(R). For I R, we set UR (I) = U(I) ∩ Min(R). When I = {r} we instead write UR (r). We also set V R (I) = Min(R) \ UR (I) and V R (r) = Min(R) \ UR (r). Thus, the collection {UR (r)}r∈R is a base for the hull-kernel topology on Min(R). Recall the following properties.

Lemma 1.3. Let R be a reduced ring and a, b ∈ R. Then
(1) UR (a) ∩ UR (b) = UR (ab);
(2) UR (a) ∪ UR (b) = UR (aR + bR);
(3) UR (a) = ∅ if and only if a = 0;
(4) UR (a) = Min(R) if and only if a is not a zero-divisor.

By (2) of Lemma 1.3 (and induction) it follows that the collection
V R (I): I R is a finitely generated ideal of R is closed under finite intersections. As a result, this collection serves as the foundation for the inverse topology, which is a topology based on Min(R). We will write Min(R)1 when we have the inverse topology. The best we can say about the collection V R (r)rR is that it is a subbase for the inverse topology in general. The reader is urged to read [10] for more information and a full description of inverse topology. We need remember the following facts about Min(R) and Min(R)1.
A zero-dimensional space is one which has a base of clopen subsets. The hull-kernel topology on Min(R) is always a zero-dimensional Hausdorff topology. In particular, it follows from Lemma 1.2 that UR (r) = V R (AnnR (r)) for any r ∈ R and so each basic open is clopen. The inverse topology makes Min(R)−1 into a compact T1-space (see Theorem 3.1 of [10]).

Definition 1.4. A ring R is said to satisfy the annihilator condition (or a.c. for short) if for all a, b ∈ R there exists a c ∈ R such that AnnR (a, b) = AnnR (c). This has obvious cardinal generalizations. For a fixed cardinal, say κ, R is said to satisfy the κ-annihilator condition if for every subset of R of cardinality less than κ, say X ⊆ R, there is an rX ∈ R such that AnnR (X) = AnnR (rX). In this sense the annihilator condition is equivalent to the ℵ0-annihilator condition. When R satisfies the κ-annihilator condition for all κ (or equivalently for |R| +), we say R satisfies the super annihilator condition.
A Baer ring is a ring R that has the property that for every subset X ⊆ R there is an idempotent e X ∈ R such that AnnR (X) = eX R. Notice then that AnnR (X) = AnnR (1 − eX) and so a Baer ring satisfies the super annihilator condition. A ring is called a weak Baer ring if for every a ∈ R there is an idempotent e ∈ R such that AnnR (a) = AnnR (e). It follows that a Baer ring is a weak Baer ring, and a weak Baer ring satisfies the annihilator condition. Weak Baer rings are also known as Rickart rings as well as p.p. rings.
Interestingly, the annihilator condition is related to the inverse topology in the following way.

Proposition 1.5. Let R be a reduced ring. R satisfies the annihilator condition if and only if the collection {V R (r)}r∈R is closed under finite intersection. In this case, the collection is a base for the inverse topology.

Proof. We first assume that R satisfies the a.c. Let r1,r2 ∈ R for which V R (r1) ∩ V R (r2) = ∅. Choose r ∈ R such that AnnR (r) = AnnR (r1,r2) = AnnR (r1) ∩ AnnR (r2).
Consider P ∈ V R (r). By Lemma 1.1 AnnR (r) P. Hence, there exists x ∈ R \ P such that xr = 0. So xr1 = xr2 = 0. Since x ∈/ P, then r1 ∈ P and r2 ∈ P, whence P ∈ V R (r1) ∩ V R (r2).
As for the reverse inclusion, let Q ∈ V R (r1) ∩ V R (r2). Then there exist x, y ∈ R \ Q such that xr1 = yr2 = 0; observe that xy ∈ R \ Q. The equation AnnR (r1,r2) = AnnR (r) yields that (xy)r = 0, and so r ∈ Q. Consequently, V R (r1) ∩ V R (r2) ⊆ V R (r).
Next, suppose {V R (r)}r∈R is closed under finite intersections. Let r1,r2 ∈ R and consider AnnR (r1,r2). By assumption there is an r ∈ R such that V R (r) = V R (r1) ∩ V R (r2). We claim AnnR (r1,r2) = AnnR (r).
Let x ∈ AnnR (r1,r2). Then xr1 = xr2 = 0. Suppose by way of contradiction that xr = 0. Since R is reduced there exists P ∈ Min(R) with xr ∈/ P giving r, x ∈/ P. Therefore, P ∈/ V R (r) = V R (r1) ∩ V R (r2). Consequently, r1 ∈/ P or r2 ∈/ P. Either way x ∈ P, a contradiction. Accordingly, AnnR (r1,r2) ⊆ AnnR (r).
On the other hand, let x ∈ AnnR (r). We claim xr1 = xr2 = 0. Suppose by way of contradiction xr1 = 0. Using that R is reduced again there exists P ∈ Min(R) with xr1 ∈/ P. So, x,r1 ∈/ P, hence P ∈/ V R (r1). Thus P ∈/ V R (r1)∩ V R (r2) = V R (r). However, xr = 0 and x ∈/ P forces r ∈ P. This contradiction leads us to conclude that x ∈ AnnR (r1,r2). We have thus demonstrated 1804 P. Bhattacharjee *et al.* / Topology and its Applications 158 (2011) 1802–1814 that AnnR (r1,r2) = AnnR (r), whence R satisfies the a.c. Consequently, a reduced ring R satisfies the a.c. precisely when the collection {V R (r)}r∈R is closed under finite intersection.

Remark 1.6. In Section 3 we will supply an example of a ring R for which the collection {V R (r): r ∈ R} forms a base for the topology on Min(R)−1 yet R does not satisfy the a.c.
Our next result was first proved by Mewborn [13] in the context of embedding a ring R inside its maximal ring of quotients (in the sense of Utumi) and determining when Min(R) is compact. In a more general setting it is stated as an exercise at the end of Section 1.6 of [9].

Proposition 1.7. Let R → S be an extension of reduced rings. For each P ∈ Spec(R) there exists Q ∈ Min(S) such that Q ∩ R ⊆ P. Furthermore, if P ∈ Min(R), there exists Q ∈ Min(S) such that P = Q ∩ R.

Definition 1.8. From this point on we will be interested in those extensions of rings, say R → S, which have the property that for each P ∈ Min(S), P ∩ R ∈ Min(R). We shall refer to such an extension as an m-extension of rings. In [16] these extensions are called minimalisant while in [17] they are called min extensions. When R → S is an m-extension we let Ψ: Min(S) → Min(R) be the map defined by Ψ (P) = P ∩ R. We continue with some examples.
First, notice that if S is a domain, then S is trivially an m-extension of any of its subrings.
Second, consider the ring homomorphism f: R → R × R which maps r to (r,r). This is an m-extension as every minimal prime ideal of R × R, say Q, has one of the forms P × R or R × P for some P ∈ Min(R). Then f −1(Q) = P ∈ Min(R).
Third, the embedding of a ring R into its classical ring of quotients, which we denote by q(R), is always an m-extension. Letting Q (R) denote the complete (a.k.a. maximal) ring of quotients of R, then it is a well-known theorem that R → Q (R) is an m-extension precisely when Min(R) is compact (see Theorem 4.3 [8] for more equivalences).

Remark 1.9. Recall from [9] that an extension of rings, say R → S is called an INC-extension if whenever Q1, Q2 ∈ Spec(S) are different primes for which Q1 ∩ R = Q2 ∩ R, then Q1, Q2 are incomparable. One might think that an INC-extension is an m-extension, however this is not the case. For any reduced ring R, the extension R → Q (R) is an INC-extension, trivially, as Q (R) is a von Neumann regular ring and therefore distinct primes are incomparable. However, we shall see later that there are examples of reduced rings R for which R → Q (R) is not an INC-extension. At this point we do not know whether a Going-Up extension (or specifically, an integral extension) must be an m-extension.

Theorem 1.10. The extension of reduced rings R → S is an m-extension if and only if whenever P ∈ Min(S) and r ∈ P ∩ R, then there exists an a ∈ R \ P such that ra = 0.

Proof. The proof follows from Lemma 1.2.

Proposition 1.11. Suppose R → S is an m-extension of reduced rings. Then Ψ: Min(S) → Min(R) is continuous with respect to both the hull-kernel topologies and the inverse topologies.

Proof. Let I be a finitely generated ideal of R and let r ∈ R. Then.

Ψ −1 V R (I) = P ∈ Min(S): I ⊆ P = V S (I)

And

Ψ −1 UR (r) = P ∈ Min(S): r ∈/ P = US (r).

Since I is finitely generated in R it generates a finitely generated ideal of S. Since the inverse image of a basic open set is again a basic open set it follows that Ψ is simultaneously continuous with respect to both the hull-kernel topologies and inverse topologies.

Remark 1.12. In [4], the author investigated exoteric extensions; in [16] they are called faiblement de Baer. We recall that a ring homomorphism f: R → S is called exoteric if whenever I, J R are finitely generated ideals of R for which AnnR (I) = AnnR (J), then AnnS (f (I)) = AnnS (f (J)). Our next proposition states that an m-extension is an exoteric homomorphism. Its proof was originally given as Proposition 24 of [16]. We include a slightly different and more straightforward proof of this fact, for completeness sake.

Proposition 1.13. If R → S is an m-extension of reduced rings, then it is an exoteric homomorphism.

Proof. Let I, J be finitely generated ideals of R such that AnnR (I) = AnnR (J). Suppose that AnnS (I) = AnnS (J). Without loss of generality, we can choose s ∈ AnnS (I) \ AnnS (J). This means there is some j ∈ J such that sj = 0. Since S is reduced, there exists P ∈ Min(S) for which sj ∈/ P; this means s ∈/ P and j ∈/ P. By the hypothesis, P ∩ R ∈ Min(R), and j ∈/ P ∩ R. Consequently, by Lemma 1.2, AnnR (J) ⊆ P ∩ R, and so AnnR (I) ⊆ P ∩ R. On the other hand, s ∈/ P implies that AnnS (s) ⊆ P. Since s I = 0, this means I ⊆ P ∩ R ∈ Min(R). Hence, AnnR (I) P ∩ R, which is a contradiction.

We conclude the introduction of the article by pointing out that we use ⊆ to mean subset or equal, while ⊂ is meant in the strict sense.

2. Extensions of commutative reduced rings

Let R → S be an extension of rings. To make things easier we let

Ψ: Min(S) → Spec(R)

be the contraction map defined by Ψ (P) = P ∩ R. Of course R → S is an m-extension precisely when Ψ (Min(S)) = Min(R). We are interested in determining when an extension of R preserves the topological properties of hull-kernel and inverse topologies. To that end we define several important kinds of extensions.

(i) S is a rigid extension of R if for each s ∈ S there is an a ∈ R such that AnnS (s) = AnnS (a).

(ii) S is an r-extension of R if for each P ∈ Min(S) and each s ∈ S \ P there exists an a ∈ R \ P such that AnnS (s) ⊆ AnnS (a).

(iii) S is an r∗-extension of R if for each P ∈ Min(S) and each s ∈ P there exists an a ∈ R ∩ P such that AnnS (a) ⊆ AnnS (s).

If in the above definitions one replaces the term a with a finitely generated ideal of R, then one gets the notions of quasi rigid, quasi r, and quasi r∗-extensions.

Example 2.1. The most common example of a rigid extension is R → q(R). In fact, any overring R → S → q(R) is a rigid extension. Furthermore, if R → T is a rigid extension and R → S → T, then both R → S and S → T are rigid extensions. The converse is true, namely the notion of rigidity is transitive. The same holds for (quasi) r- and (quasi) r∗-extensions.

Remark 2.2. In [1] and [2] the author investigated when an extension of frames is rigid. We prefer not to delve into these matters but we do point out the collection of all radical ideals of R, denoted Rad(R) forms a frame and in certain nice situations the extension R → S can be characterized as a rigid extension in terms of the rigidity of the extensions of the Rad(R) → Rad(S).

Proposition 2.3. A (quasi) rigid extension is both a (quasi) r-extension and a (quasi) r∗-extension. Proof. We supply a proof that a rigid extension is an r-extension and an r∗-extension. We leave the case for quasi rigid extensions to the interested reader.

For each s ∈ S choose rs ∈ R for which AnnS (rs) = AnnS (s). Let P ∈ Min(S). For s ∈ S \ P, then AnnS (rs) = AnnS (s) P. By Lemma 1.2, r ∈/ P. On the other hand if s ∈ P, then by Lemma 1.1 AnnS (rs) = AnnS (s) P. Thus, rs ∈ P.

Proposition 2.4. Suppose R → S is a rigid extension. R satisfies the κ-annihilator condition if and only if S satisfies the κ-annihilator condition.

Proof. We prove the sufficiency and leave the proof of the necessity to the interested reader. Let X ⊆ R of cardinality smaller than κ and consider AnnS (X). Since S satisfies the κ-annihilator condition there is an element e ∈ S such that AnnS (X) = AnnS (e). Since R → S is a rigid extension there is an r ∈ R such that AnnS (e) = AnnS (r). From here we leave it to the interested reader to check that AnnR (X) = AnnR (r).

Corollary 2.5. R satisfies the a.c. if and only if q(R) satisfies the a.c.

The proof of the next lemma is straightforward and is left to the interested reader.

Lemma 2.6. Suppose R → S is a quasi rigid extension. If S satisfies the κ-annihilator condition then for each subset $X \subseteq R$ of cardinality smaller than κ there is a finite subset $Y \subseteq R$ such that $\mathrm{AnnR}(X) = \mathrm{AnnR}(Y)$.

Our next two results demonstrate the importance of r-extensions.

Lemma 2.7. Let $R \to S$ be an extension of reduced rings. If S is an r-extension of R, then Ψ is a bijection of Min(S) onto Min(R). In particular, an r-extension is an m-extension.

Proof. Let $P \in \mathrm{Min}(S)$. We first show that $P \cap R \in \mathrm{Min}(R)$. Otherwise, there is a $Q \in \mathrm{Min}(R)$ with $Q \subset P \cap R$. Now by Proposition 1.7 there exists $M \in \mathrm{Min}(S)$ with $M \cap R = Q$. Choose $s \in M \setminus P$. By hypothesis there exists an $r \in R \setminus P$ with $\mathrm{AnnS}(s) \subseteq \mathrm{AnnS}(r)$. Since $s \in M$, it follows that AnnS (s) M, whence AnnS (r) M. Thus, $r \in M \cap R = Q$. But $Q \subseteq P$ and so $r \in P$, a contradiction. Therefore Ψ: $\mathrm{Min}(S) \to \mathrm{Min}(R)$ is a surjective map.

Now we show Ψ is an injection. Take distinct P and Q in Min(S) and choose $s \in P \setminus Q$. By hypothesis there is an $r \in R \setminus Q$ for which $\mathrm{AnnS}(s) \subseteq \mathrm{AnnS}(r)$. A similar argument to the one just used yields that $r \in P$. Therefore, $r \in P \cap R$, but $r \in/ Q \cap R$. Hence $P \cap R = Q \cap R$.

Theorem 2.8. Let $R \to S$ be an extension of reduced rings. The following statements are equivalent.

(1) S is an r-extension of R.

(2) S is a quasi r-extension of R.

(3) Ψ: $\mathrm{Min}(S) \to \mathrm{Min}(R)$ is a homeomorphism (with regards to the hull-kernel topology).

Proof. The proof of the implication that (1) implies (2) is patent.

Suppose that S is a quasi r-extension of R. To show that S is an r-extension of R let $P \in \mathrm{Min}(S)$ and $s \in S \setminus P$. Choose I, a finitely generated ideal of R, so that I P and $\mathrm{AnnS}(s) \subseteq \mathrm{AnnS}(I)$. Choose $a \in I \setminus P$ and observe that $\mathrm{AnnS}(I) \subseteq \mathrm{AnnS}(a)$. Thus, (2) implies (1).

Next, suppose that S is an r-extension of R. We aim to show that Ψ: $\mathrm{Min}(S) \to \mathrm{Min}(R)$ is a homeomorphism (with regards to the hull-kernel topology). By Lemma 2.7 and Proposition 1.11 it suffices to show that Ψ is an open map. Let $s \in S$ and consider Ψ (US (s)). Note that

Ψ US (s) = $P \cap R$: $P \in \mathrm{Min}(S)$, $s \in/ P$.

Let $P \cap R \in \Psi$ (US (s)), and so since Ψ is a bijection we gather that $s \in/ P$. Since $R \to S$ is an r-extension there is an a $\in R \setminus P$ with $\mathrm{AnnS}(s) \subseteq \mathrm{AnnS}(a)$. That $a \in/ P \cap R$ means that $P \cap R \in$ UR (a). We claim that UR (a) $\subseteq \Psi$ (US (s)) which yields that Ψ (US (s)) is an open subset (relative to the hull-kernel topology).

Take $M \in$ UR (a) and choose $Q \in \mathrm{Min}(S)$ with $Q \cap R = M$. Since $a \in/ M$ it happens that $a \in/ Q$ and thus $\mathrm{AnnS}(a) \subseteq Q$. By our choice of a, $\mathrm{AnnS}(s) \subseteq \mathrm{AnnS}(a) \subseteq Q$, we may apply Lemma 1.2 to conclude that $s \in/ Q$, i.e. $Q \in$ US (s). Therefore, $M = Q \cap R \in \Psi$ (US (s)), whence Ψ is an open map. Thus, (1) implies (3).

Finally, suppose that Ψ: $\mathrm{Min}(S) \to \mathrm{Min}(R)$ is a homeomorphism (with regards to the hull-kernel topology). Let $P \in \mathrm{Min}(S)$ and let $s \in S \setminus P$. Then $P \in$ US (s). By hypothesis Ψ (US (s)) is an open subset of Min(R), and thus there is an $r \in R$ such that Ψ (P) $\in$ UR (r) $\subseteq \Psi$ (US (s)). That Ψ (P) $\in$ UR (r) means that $P \cap R \in$ UR (r) and so $r \in/ P$.

Next, we demonstrate that $\mathrm{AnnS}(s) \subseteq \mathrm{AnnS}(r)$. Let $a \in \mathrm{AnnS}(s)$ and consider $ra \in S$. If $Q \in$ V S (r), then $ra \in Q$. If $Q \in$ US (r), then $r \in/ Q$, and so $Q \cap R \in$ UR (r) $\subseteq \Psi$ (US (s)). It follows that $s \in/ Q$ and so $a \in Q$. Consequently, $ra \in Q$ for all $Q \in \mathrm{Min}(S)$. Since we are assuming that S is a reduced ring, by Lemma 1.2 it follows that $ra = 0$, whence $a \in \mathrm{AnnS}(r)$. Thus, (3) implies (1).

Theorem 2.9. Suppose $R \to S$ is an r-extension of reduced rings. $R \to S$ is a rigid extension if and only if Ψ maps basic open sets to basic open sets (with respect to the hull-kernel topologies).

Proof. Suppose $R \to S$ is a rigid extension. Then for each $s \in S$ there exists $rs \in R$ such that AnnS (rs) = AnnS (s). We leave it to the interested reader to check that Ψ (US (s)) = UR (rs). From this we conclude that Ψ maps basic open sets to basic open sets.

Conversely, let $s \in S$ and consider US (s). By hypothesis, there exists $r \in R$ such that Ψ (US (s)) = UR (r). Therefore, Ψ (V S (s)) = V R (r). We now demonstrate that AnnS (r) = AnnS (s).

Let $P \in \mathrm{Min}(S)$ be arbitrary. If $P \in$ US (s), then $P \cap R \in$ UR (r). Therefore, both AnnS (s) and AnnS (r) are subsets of P; hence r AnnS (s) $\subseteq P$ and s AnnS (r) $\subseteq P$. Similarly, if $P \in$ V S (s), then $P \cap R \in$ V R (r). So, both $s \in P$ and $r \in P$. Hence, $r \cdot \mathrm{AnnS}(s) \subseteq P$ and $s \cdot \mathrm{AnnS}(r) \subseteq P$. Since the rings are reduced, this implies that

$r \cdot \mathrm{AnnS}(s) = 0 = s \cdot \mathrm{AnnS}(r)$.

Consequently AnnS (s) = AnnS (r), concluding that the extension is a rigid extension.

We now turn to quasi r$*$-extensions and prove some analogous results. We shall have several occasions to use our next lemma.

Lemma 2.10. Let $R \to S$ be an extension of reduced rings. For $P \in \mathrm{Min}(S)$ and I a finitely generated ideal of R. If $I \subseteq P \cap R$, then AnnS (I) P. The converse is also true.

Proof. Let J = I S be the ideal in S generated by I. Since $I \subseteq P$, it follows that $J \subseteq P$. By Lemma 1.2 (J is a finitely generated ideal of S) it follows that AnnS (J) P. Also since $\mathrm{AnnS}(J) \subseteq \mathrm{AnnS}(I)$, we conclude that AnnS (I) P. The proof of the converse is a simple application of Lemma 1.2.

Proposition 2.11. A quasi r$*$-extension of reduced rings $R \to S$ is an m-extension. Moreover, the map Ψ: $\mathrm{Min}(S) \to \mathrm{Min}(R)$ is a bijection.

Proof. Let $P \in \mathrm{Min}(S)$. Since $P \cap R \in \mathrm{Spec}(R)$, there exists some $Q \in \mathrm{Min}(R)$ such that $Q \subseteq P \cap R$. By Proposition 1.7 there exists $M \in \mathrm{Min}(S)$ such that $M \cap R = Q$. If $M = P$ then we can choose $s \in M \setminus P$. By hypothesis there exists a finitely generated ideal $I \subseteq M \cap R$ such that $\mathrm{AnnS}(I) \subseteq \mathrm{AnnS}(s)$. Since $I \subseteq Q \subseteq P$, it follows from the preceding lemma that AnnS (I) P. However, $\mathrm{AnnS}(s) \subseteq P$ since $s \in/ P$, which is a contradiction. Thus, $M = P$, whence $P \cap R = Q \in \mathrm{Min}(R)$. Consequently, $R \to S$ is an m-extension.

To show that Ψ is bijective it suffices to show that it is injective. To that end consider two distinct minimal prime ideals $P, Q \in \mathrm{Min}(S)$. Choose $s \in P \setminus Q$. Again, using the hypothesis that $R \to S$ is a quasi r$*$-extension, there is a finitely generated ideal $I \subseteq P \cap R$ such that AnnS (I) $\subseteq$ AnnS (s). Notice that AnnS (I) $\subseteq Q$. Therefore, by the preceding lemma, $I\ Q \cap R$. Let $r \in I \setminus Q$, then $r \in (P \cap R) \setminus (Q \cap R)$. Hence, $P \cap R = Q \cap R$.

Theorem 2.12. Suppose $R \to S$ is an extension of reduced rings. S is a quasi r∗-extension of R if and only if Ψ: $\mathrm{Min}(S)^{-1} \to \mathrm{Min}(R)^{-1}$ is a homeomorphism.

Proof. Necessity: We have already shown in Proposition 2.11 that Ψ is a bijection. Moreover, by Proposition 1.11 Ψ is a continuous map with respect to the inverse topologies. It remains to show that Ψ is an open map. Let $s \in S$ and $Q \in \Psi$ (V S (s)). There exists some $P \in$ V S (s) with $Q = P \cap R$. So, $s \in P$. By the definition of quasi r∗-extension there exists a finitely generated ideal I R such that $I \subseteq P \cap R$ and AnnS (I) $\subseteq$ AnnS (s). Clearly, $Q \in$ V R (I). We now show that if $M \in$ V R (I), then $M \in \Psi$ (V S (s)). Choose $T \in \mathrm{Min}(S)$ for which $T \cap R = M$. Since $I \subseteq M$, then $I \subseteq T$ and so AnnS (I) T. Furthermore, AnnS (s) T which forces $s \in T$. Therefore, $T \in$ V S (s) and so $M = T \cap R \in \Psi$ (V S (s)). It follows that Ψ (V S (s)) is an open set, whence Ψ is an open map.

Sufficiency: Let $P \in \mathrm{Min}(S)$ and $s \in P$. Since V S (s) is a open subset of $\mathrm{Min}(S)^{-1}$ and Ψ is an open map, it follows that Ψ (V S (s)) is open in $\mathrm{Min}(R)^{-1}$. So, $P \cap R \in \Psi$ (V S (s)). Therefore, there exists a finitely generated ideal I of R such that $P \cap R \in$ V R (I) $\subseteq \Psi$ (V S (s)); thus $I \subseteq P \cap R$. We need to demonstrate that AnnS (I) $\subseteq$ AnnS (s). Let $t \in$ AnnS (I) and consider $st \in S$. Let $Q \in \mathrm{Min}(S)$ be an arbitrary minimal prime ideal of S. If $s \in Q$, then $st \in Q$. On the other hand, if $s \notin Q$, then $Q \cap R \notin \Psi$ (V S (s)); observe that we are assuming that Ψ is a bijection. Therefore, $Q \cap R \notin$ V R (I), i.e. $I\ Q \cap R$. Then I Q, and so AnnS (I) $\subseteq Q$. Consequently, $t \in Q$, and thus $st \in Q$. Since Q was arbitrarily chosen and S is assumed to be reduced we conclude that $st = 0$, i.e. $t \in$ AnnS (s).

Proposition 2.13. If $R \to S$ is a quasi r∗-extension of reduced rings, then $R \to S$ is a quasi rigid extension if and only if Ψ maps inverse basic open sets to inverse basic open sets.

Proof. Suppose $R \to S$ is a quasi rigid extension. In order to show that Ψ maps inverse basic open sets to inverse basic open sets, let $J = s_1 S + \cdots + s_n S$ be an arbitrary finitely generated ideal of S. By hypothesis, for each $i = 1,\dots,n$ there exists a finitely generated ideal of R, say I_i, such that AnnS (I_i) = AnnS (s_i). Set $I = I_1 + \cdots + I_n$. We claim that Ψ (V S (J)) = V R (I). Let $P \in$ V S (J), i.e. $J \subseteq P$. This means that $s_1,\dots, s_n \in P$. Therefore, by Lemma 2.10 AnnS (s_i) P, implying that AnnS (I_i) P for all i. Therefore, $I_i \subseteq P$ for each i. Thus $I \subseteq P \cap R$, whence $P \cap R \in$ V R (I). This means that Ψ (V S (J)) $\subseteq$ V R (I). On the other hand, if $P \cap R \in$ V R (I), then $I \subseteq P$ and so $I_i \subseteq P$ for all i. Therefore, AnnS (I_i) P for all i. So for all i, AnnS (s_i) P, which says that $s_i \subseteq P$. Consequently, $J \subseteq P$, that is, $P \cap R \in \Psi$ (V S (J)). This means that the reverse containment V R (I) $\subseteq \Psi$ (V S (J)) also holds.

Conversely, suppose that Ψ maps inverse basic open sets to inverse basic open sets and let $s \in S$. By the hypothesis there exists I R a finitely generated ideal such that Ψ (V S (s)) = V R (I). Therefore, Ψ (US (s)) = UR (I). We claim that AnnS (I) = AnnS (s). Using a similar argument as in the preceding theorem (replace r by I), it follows that

$I \cdot$ AnnS (s) $= 0 = s \cdot$ AnnS (I).

Hence AnnS (I) = AnnS (s), which implies that the extension is a quasi rigid extension.

Next, we consider what happens when R satisfies the a.c.

Theorem 2.14. Let $R \to S$ be an extension of reduced rings and suppose that R satisfies the a.c. The following statements are equivalent.

(1) $R \to S$ is a quasi r∗-extension.

(2) Ψ: $\mathrm{Min}(S)^{-1} \to \mathrm{Min}(R)^{-1}$ is a homeomorphism.

(3) $R \to S$ is an r∗-extension.

Proof. That (1) and (2) are equivalent is Theorem 2.12. Clearly, (3) implies (1).

Suppose $R \to S$ is a quasi r∗-extension and that R satisfies the a.c. By Proposition 1.5 {V R (r): $r \in R$} is a base for the inverse topology on $\mathrm{Min}(R)^{-1}$. To show that the extension in an r∗-extension, let $P \in \mathrm{Min}(S)^{-1}$ and $s \in P$. Since Ψ is an open map, Ψ (V S (s)) is open in $\mathrm{Min}(R)^{-1}$ Thus, there exists some $r \in R$ such that $P \cap R \in$ V R (r) $\subseteq \Psi$ (V S (s)); $r \in P \cap R$. We aim to show that AnnS (r) $\subseteq$ AnnS (s); let $t \in$ AnnS (r). Observe that for any $Q \in \mathrm{Min}(S)^{-1}$ if $s \notin Q$, then $r \notin Q$ because Ψ is injective. Therefore, $t \in Q$ and hence so is $st \in Q$. In the case that $s \in Q$ then so is st. Since S is reduced it follows that $ts = 0$. Consequently, $R \to S$ is an r∗-extension.

We end this section by considering the extension $R \to R[x]$. First, a useful lemma.

Lemma 2.15. Suppose $R \to S$ is a quasi rigid extension and R satisfies the a.c. Then the extension is a rigid extension.

Proof. Let $s \in S$ and choose $r_1,\dots,r_n \in R$ such that

AnnS (s) = AnnS ($r_1,\dots,r_n$).

Next, choose $r \in R$ such that

AnnR ($r_1,\dots,r_n$) = AnnR (r).

We aim to prove that AnnS (s) = AnnS (r).

Let $x \in$ AnnS (s). Choose $a_1,\dots,a_m \in R$ such that

AnnS (x) = AnnS ($a_1,\dots,a_m$).

Then $sa_1 = \cdots = sa_m = 0$ and so $a_1,\dots,a_m \in$ AnnS (s) = AnnS ($r_1,\dots,r_n$). It follows that each $a_i \in$ AnnR ($r_1,\dots,r_n$) = AnnR (r). Therefore,

$r \in$ AnnS ($a_1,\dots,a_m$) = AnnS (x).

In other words, $x \in$ AnnS (r). Therefore, AnnS (s) $\subseteq$ AnnS (r).

Next, let $x \in$ AnnS (r) and suppose AnnS (x) = AnnS ($a_1,\dots,a_m$) for $a_1,\dots,a_m \in R$. So $r \in R \cap$ AnnS ($a_1,\dots,a_m$) = AnnR ($a_1,\dots,a_m$). Switching it around, $a_1,\dots,a_m \in$ AnnS ($r_1,\dots,r_n$) = AnnS (s). Therefore, $s \in$ AnnS ($a_1,\dots,a_m$) = AnnS (x). Therefore, $x \in$ AnnS (s).

Proposition 2.16. Suppose R is a reduced ring. The extension $R \to R[x]$ is a quasi rigid extension (and therefore an m-extension).

Proof. Let $f(x) \in R[x]$ and set $f(x) = a_0 + a_1x + \cdots + a_nx^n$. Set $I = Ra_0 + \cdots + Ra_n$, a finitely generated ideal of R. By Proposition B of [5],

AnnR[x] f (x) = AnnR (I) [x].

It follows that if $g(x) \in$ (AnnR (I))[x], then $g(x)a_k = 0$ for each $k = 0,\dots,n$, whence (AnnR (I))[x] $\subseteq$ AnnR[x](a_k) for each k. Conversely, if $a_k g(x) = 0$ for each k, then every coefficient of g(x) annihilates each a_k. Therefore, AnnR[x] f (x) = AnnR (I) [x] = AnnR[x]($a_0,\dots,a_n$).

Corollary 2.17. Suppose R is a reduced ring and $x_1, x_2,\dots, x_n$ is a finite number of indeterminates. Then $R \to R[x_1,\dots, x_n]$

is a quasi rigid extension.

Proposition 2.18. Suppose R is a reduced ring. R → R[x] is a rigid extension if and only if R satisfies the a.c. **Proof.** The proof of the statement consists of a simple application of Proposition 2.4 and Lemma 2.15, together with the known fact (see Corollary 2.9 of [8]) that R[x] always satisfies the a.c. when R is a reduced ring.

Remark 2.19. Observe that if R is reduced ring that does not satisfy the a.c., then R → R[x] is an example that shows we cannot generalize Proposition 2.4 to quasi rigid extensions

Compact minimal prime spectra

We discussed rigid, r-, and r-extensions in the previous section, as well as the links between them (and their quasi counterparts). We also had multiple opportunities to think about what occurs if the base ring R is assumed to satisfy the a.c. When our intended extension fulfils the additional hypothesis that either Min(R) or Min(S) is compact, we'll look at this section. We'll start by recalling a useful theorem that characterises this situation.

Proposition 3.1. (Proposition 3.2 [10]) For a reduced ring R, the following statements are equivalent.

(1) Min(R) is compact.

(2) Min(R) = Min(R)−1.

(3) For each a ∈ R there exists a finitely generated ideal I R, such that I AnnR (a) and AnnR (aR + I) = 0.

We remark that when we write Min(R) = Min(R)−1 we mean that the hull-kernel and inverse topologies are the same. Since the hull-kernel topology is finer than the inverse this is also equivalent to saying that the inverse topology generates the hull-kernel topology. If R is a ring for which Min(R) is compact yet R does not satisfy the a.c., then the collection {V R (r): r ∈ R} forms a base for the topology while not being closed under intersection. This addresses Remark 1.6.

To give a flavor of the style of theorems we aim to prove we next demonstrate that when Min(R) is compact then Proposition 1.13 can be strengthened.

Proposition 3.2. (Proposition 24 [16]) Let R → S be an extension of reduced rings and suppose that Min(R) is compact. R → S is an exoteric homomorphism if and only if it is an m-extension.

Proposition 3.3. Suppose that R is reduced and Min(R) is compact. For any extension of reduced rings, say R → S, the extension is an r-extension if and only if it is a quasi-rigid extension.

Proof. To prove the sufficiency, recall that a quasi-rigid extension is a quasi r-extension. Theorem 2.8 states that a quasi r-extension is an r-extension. Therefore, a quasi-rigid extension is an r-extension.

Conversely, suppose that Min(R) is compact and that R → S is an r-extension of reduced rings. Let s ∈ S. By Theorem 2.8, ψ: Min(S) → Min(R) is a homeomorphism and thus an open map. So Ψ (US (s)) ⊆ Min(R) is a closed, and hence compact, open subset. Therefore, there is a finite set r1,...,rn ∈ R such that

Ψ US (s) = UR (r1) ∪···∪ UR (rn) = UR (I)

where I is the ideal generated by r1,...,rn. We claim that AnnS (s) = AnnS (I).

First, let t ∈ AnnS (I). By means of contradiction assume that t ∈/ AnnS (s), i.e. ts = 0. Since S is reduced there is a minimal prime ideal P ∈ Min(S) such that ts ∈/ P. It follows that P ∈ US (s) and so P ∩ R ∈ Ψ (US (s)) = UR (I). This means that I P, whence AnnS (I) P. Since t ∈ AnnS (I), we conclude that t ∈ P, hence st ∈ P, contradicting that st ∈/ P. This contradiction forces t ∈ AnnS (s). Since t was arbitrarily chosen we conclude that AnnS (I) ⊆ AnnS (s).

To show the reverse containment let t ∈ AnnS (s), and once again assume, by means of contradiction, that t ∈/ AnnS (I). It follows that for some i = 1,...,n tri = 0. Since S is reduced there is some P ∈ Min(S) such that tri ∈/ P, thus t ∈/ P. Next, since ts = 0, we gather that s ∈ P, whence P ∈/ US (s). By Theorem 2.8 Ψ is a bijection and so P ∩ R ∈/ Ψ (US (s)) = UR (I). Equivalently, this last statement means that I ⊆ P ∩ R. Therefore, ri ∈ P and so tri ∈ P, yielding the desired contradiction. We are forced to conclude that t ∈ AnnS (I), and since t was arbitrary we obtain the reverse containment AnnS (s) ⊆ AnnS (I).

Corollary 3.4. Let S be a reduced ring for which Min(S) is compact. For any subring R of S, the extension R → S is an r-extension if and only if it is a quasi rigid extension.

Proof. If the extension is an r-extension, then Min(S) is homeomorphic to Min(R). Hence, the hypothesis implies Min(R) is compact and so Proposition 3.3 applies.

Proposition 3.5. Let S be a reduced ring for which Min(S) is compact. For any subring R of S, the extension R → S is an (quasi) r∗-extension if and only if it is a (quasi) rigid extension.

Proof. First of all we recall that Proposition 2.3 states that in general a rigid extension is an r∗-extension. Next, we supply a proof that whenever Min(S) is compact then an r∗-extension is a rigid extension, leaving the quasi case for the interested reader. Let s ∈ S and suppose that R → S is an r∗-extension. For each P ∈ V S (s) choose rP ∈ P ∩ R such that AnnS (rP) ⊆ AnnS (s). It follows that the collection {V S (rP): P ∈ V S (s)} is an open cover of V S (s). Since Min(S) is compact V S (s) is a compact open subset and so there is a finite collection rP1,...,rPn ∈ P ∩ R such that

V S (s) = V S (rP1) ∪···∪ V S (rPn) = V S (rP1 ···rPn).

It follows that AnnS (s) = AnnS (rP1 ···rPn), whence R → S is a rigid extension.

Two standard examples of classes of rings whose minimal prime ideal spaces are comp1act are the Baer ring and von Neumann regular rings. This leads to our next corollary.

Corollary 3.6. A Baer ring is an r∗-extension if and only if it is a rigid extension of any of its subrings. The same is true for a von Neumann regular ring.

Theorem 3.7. Let R → S be an extension of reduced rings. Suppose that Min(S) is compact and R satisfies the a.c. The following are equivalent.

(1) S is an r-extension of R.

(2) S is an r∗-extension of R.

(3) S is a rigid extension of R.

Proof. That (2) and (3) are equivalent and that (3) implies (1) follows from Proposition 2.3 and Lemma 3.4. To get that (1) implies (3) use Proposition 3.3 and Lemma 2.15.

Example 3.8. Recall that if X is a topological space, C(X) (resp. C(X,Z)) denotes the ring of continuous real valued (resp. integer-valued) functions on X. In Example 3.18 of [1], the author proved that if X is a compact zero-dimensional F - space, then C(X,Z) C(X) is an r∗-extension. Furthermore, this extension is a rigid extension precisely when X is basically disconnected. What is interesting is that in this case Min(C(X,Z)) is homeomorphic to X and hence is compact, however in general Min(C(X)) is not compact.
Now, the space βN \ N is a compact zero-dimensional F - space which is not basically disconnected. It follows that
C(βN \ N,Z) C(βN \ N) is an example of an r∗-extension which is not a rigid extension. This shows that Proposition 3.5 cannot be generalized to the case that Min(R) is compact.

3. Essential extensions

The characterization of epimorphisms (morphisms f for which m f = n f implies m = n) in a category is a crucial task. In many categories, epimorphisms are exact surjective maps; however, this is not the case in the category of commutative rings. The embedding of a ring into its classical ring of quotients is a typical example of a non-surjective epimorphism. Isbell began studying algebraic epimorphisms in earnest in a series of papers titled Epimorphisms and dominions. Storrer [21] deserves a lot of credit for commutative rings.
Storrer showed that for a given reduced commutative ring R there is a largest epimorphic essential extension. What this means is that for a reduced commutative ring R there exists an extension R → E(R) which is both an essential and epimorphic extension, and such that whenever R → S is another essential epimorphic extension, then there is an embedding of S into E(R) which restricts to the identity on R. Nowadays, such an extension is known as the epimorphic hull of R E(R) can be described as the intersection of the set of von Neumann regular rings lying intermediate between R and Q (R); it itself is a von Neumann regular ring. It is also described as the subring of Q (R) generated by R and the quasi-inverses of elements of R.
Since R → q(R) is an essential epimorphic extension we gather that R → q(R) → E(R). What distinguishes the extensions R → q(R) and R → E(R) is that the former is always a flat epimorphism (that is, an extension R → S for which S is a flat R-module). Several authors have been instrumental in showing that a maximal flat epimorphic essential extension exists. We let M(R) denote the maximal flat epimorphic ring of quotients of R. Observe that

R → M(R) → E(R) → Q (R).

In this section, we'll look at how rigid these extensions are. We'll also have the chance to think about the Baer hull of a reduced ring R. Remember that the intersection of all Baer subrings of Q (R) containing R is the subring of Q (R), labelled B(R), created by R and the idempotents of Q, as stated in Proposition 2.5 of [14]. (R). B(R) is the Baer hull of R since Mewborn established that B(R) is a Baer ring.

Remark 4.1 The reader is referred to [20] and [19] for a full discussion of the epimorphic hull and maximal flat epimorphic essential extension, as well as a list of references. [19] and [3] are two further references on the epimorphic hull that the reader could find valuable.

Remark 4.2. The authors of [6] looked at when the extension R Q (R) is a rigid extension for a decreased f-ring R. The language is couched in terminology from the theory of latticeordered groups and function-rings, thus reading the text requires some caution. In a sense, Theorem 4.3 is a generalisation of their theorem to the case of commutative reduced rings in general, thus it's no surprise that our proof is based on theirs with some modifications.
We first look at when R → Q (R) is a rigid extension. Recall that this extension of rings is an m-extension precisely when Min(R) is compact (Theorem 4.3 [8]). Therefore, it is necessary that Min(R) be compact for R → Q (R) to be a rigid extension. We can say more.
When R is reduced then Q (R) is a von Neumann regular Baer ring. It follows that Min(Q (R)) is a compact extremally disconnected space; a space is called extremally disconnected if the closure of every open set is clopen (Proposition 2.1) [14]). Thus, if R → Q (R) is a quasi rigid extension then Min(R) is also compact extremally disconnected by Proposition 2.3 and Theorem 2.8. Furthermore, if the extension is rigid then we also know that R satisfies the a.c. by the fact that Q (R) satisfies the a.c. and Proposition 2.4. We presently show that these properties characterize when R → Q (R) is a rigid extension.

Theorem 4.3. Let R be a reduced commutative ring with identity. The following statements are equivalent. (1) R → Q (R) is a rigid extension.
(1) R → Q (R) is an r∗-extension.
(2) R → B(R) is a rigid extension.
(2) R → B(R) is an r∗-extension.
(3) Min(R) is a compact extremally disconnected space and R satisfies the a.c.
(4) q(R) is a Baer ring.
(5) q(R) and Q (R) have the same idempotents.

Proof. That (1) and (1) are equivalent and (2) and (2) are equivalent follows from Corollary 3.6. The proof that statement (1) implies (2) is patent.
If R → B(R) is a rigid extension, then as we mentioned above, since B(R) is a Baer ring, Min(B(R)) is a compact extremally disconnected and hence by rigidity so is Min(R) (Proposition 2.3 and Theorem 2.8). Furthermore, since B(R) is a von Neumann regular ring it satisfies the a.c. and thus rigidity implies that R satisfies the a.c. by Theorem 2.4. This means that (2) implies (3).
Suppose that Min(R) is a compact extremally disconnected space and R satisfies the a.c. First, by Theorem 4.5 of [8] we gather that q(R) is a von Neumann regular ring. Moreover, Min (q(R)) is compact extremally disconnected. By Proposition 2.1 of [14] it follows that q(R) is a Baer ring. Thus, (3) implies (4).
If q(R) is a Baer ring, it follows that B(R) q(R) and so q(R) and Q (R) share the same idempotents. Hence (4) implies (5).
Finally, suppose q(R) and Q (R) have the same idempotents. Let a ∈ Q (R). Then there is some idempotent e ∈ Q (R) such that aQ (R) = eQ (R). By hypothesis e ∈ q(R) and so e = r s for some r, s ∈ R. Then it is straightforward to check that
AnnQ (R)(a) = AnnQ (R)(e) = AnnQ (R)(r) and so R → Q (R) is a rigid extension. Consequently, (5) implies (1).

It should be clear by now that the arguments in Theorem 4.3 apply with minor modifications when the rigid criterion is weakened to quasi rigid. For the purpose of thoroughness, we've included a proof of this.

Theorem 4.4. Let R be a reduced commutative ring with identity. The following statements are equivalent.
(1) R → Q (R) is a quasi rigid extension.
(2) R → B(R) is a quasi rigid extension.
(3) Min(R) is a compact extremally disconnected space.
(4) Min(q(R)) is a compact extremally disconnected space.
(5) Min(R) is compact and every annihilator ideal of R is the annihilator of a finitely generated ideal of R. **Proof.** If R → Q (R) is a quasi rigid extension, then so is R → S for any subring S of Q (R) containing R. Therefore, in this case R → B(R) is a quasi rigid extension. Thus (1) implies (2).
Suppose that R → B(R) is a quasi rigid extension. It follows from Proposition 2.3 and Theorem 2.8 that Min(R) is homeomorphic to Min(B(R)). Since B(R) is a Baer ring Min(B(R)), and hence Min(R), is a compact extremally disconnected space. Thus (2) implies (3).
It is always the case that Min(R) and Min(q(R)) are homeomorphic. Thus (3) and (4) are equivalent.
To show that (3) and (5) are equivalent we shall use of the following lemma.

Lemma 4.5. Suppose T is a reduced commutative ring with identity. For any ideal I of T the closure of U(I) in Min(T) equals V (AnnT (I)). Moreover, for any pair of ideals I and J, V (AnnT (I)) = V (AnnT (J)) if and only if AnnT (I) = AnnT (J).

Proof. Suppose P ∈ Min(T) and I P. Then AnnT (I) ⊆ P. Therefore, U(I) ⊆ V (AnnT (I)). Conversely, let P ∈ Min(T) and suppose that AnnT (I) ⊆ P. Let t ∈ T and suppose that P ∈ U(t). This means that t ∈/ P. If it were the case that U(t) ∩ U(I) = ∅, then this would mean that t I = 0 since T is assumed to be reduced. It would then follow that t ∈ AnnT (I) and hence t ∈ P. This apparent contradiction means that U(t) ∩ U(I) = ∅. Since t was arbitrarily chosen we conclude that every basic open set around P meets U(I) and so P ∈ U(I) showing the reverse containment.
As for the second statement, if V (AnnT (I)) = V (AnnT (J)) but AnnT (I) = AnnT (J), then there would exist, without loss of generality, an x ∈ AnnT (I)/AnnT (J). This means that there is some j ∈ J such that xj = 0 and so (since T is reduced) there is some P ∈ Min(T) such that xj ∈/ P, hence x ∈/ P. But then I ⊆ P, i.e. P ∈ V (I). By assumption V (I) = V (J) so that J ⊆ P, a contradiction.
Notice that if Min(R) is a compact extremally disconnected space then for any ideal I of R, U(I) = V (AnnT (I)) is a clopen subset and hence compact and open. It follows that V (AnnR (I)) = V (AnnR (J)) for some finitely generated ideal J of R (use (2) of Lemma 1.3). By the just-proved lemma AnnR (I) = AnnR (J). Conversely, if every annihilator ideal is the annihilator of a finitely generated ideal, then since any open subset of Min(R) is of the form U(I) for some ideal the lemma implies that U(I) = V (AnnR (I)). The hypothesis implies that AnnR (I) = AnnR (J) for some finitely generated ideal J of R. Since V (AnnR (J)) is clopen we conclude that Min(R) is extremally disconnected. Thus, (3) and (5) are equivalent. Finally, suppose that Min(R) is compact and that the annihilator of every ideal of R is the annihilator of a finitely generated ideal. To show that R → Q (R) is a quasi rigid extension we need to recall the construction of Q (R) (see [11]). An arbitrary element of Q (R) is an equivalence class of R-module homomorphisms defined on a dense ideal of R with values in R. Two such modules are equivalent if they agree on the product of their domains. Given [φ],[ψ] ∈ Q (R) where φ: I → R and ψ: J → R (I, J dense ideals of R), [φ][ψ]=[α] where α: I J → R is the R-module homomorphism which maps ij to φ(i)ψ(j) for all i ∈ I and j ∈ J. Given s ∈ Q (R), set s = [φ] for φ: I → R, I a dense ideal. We leave it to the interested reader to show that φ(I) is an ideal of R. By hypothesis there is a finitely generated ideal of R, say J = r1R +···+ rnR, such that AnnR (φ(I)) = AnnR (J). We claim that

AnnQ (R)(s) = AnnQ (R)(r1,...,rn)

From which it will follows that R → Q (R) is a quasi rigid extension. Let t ∈ AnnQ (R)(s) and set t = [τ] for some Rmodule homomorphism τ: Z → R where Z is a dense ideal of R. Then given [α]=[φ][τ], where α is defined as above, we observe that for all z ∈ Z and i ∈ I, 0 = α(iz) = φ(i)τ (z). Thus, τ (Z)φ(I) = 0 and so riτ (Z) = 0 for each i = 1,...,n. Thus, t ∈ AnnQ (R)(r1,...,rn). We have shown that AnnQ (R)(s) = AnnQ (R)(r1,...,rn). We leave the proof of the reverse containment to this interested reader.
Next, we analyse whether R's epimorphic hull embedding is a stiff extension. E(R) is an m-extension if and only if E(R) is a flat R-module, as E(R) is a von Neumann regular ring R. As a result, we get the following. (For other equivalences, see [8]'s Corollary 3.3 and Theorem 4.5.)

Theorem 4.6. Let R be a reduced commutative ring with identity. Then the following statements are equivalent.
(1) R → E(R) is a rigid extension.
(2) Min(R) is compact and R satisfies the a.c.
(3) q(R) is von Neumann regular.
(4) q(R) = E(R).
(5) q(R) is a weak Baer ring.

Proof. That (2) is equivalent to (3) is part of Theorem 4.5 of [8]. Since E(R) is the smallest von Neumann regular subring of Q (R) containing R and q(R) ⊆ E(R) it follows that (3) and (4) are equivalent. That (3) and (5) are equivalent is well known; one may look at the remark after Definition 2.10 of [10]. This (2), (3), (4), and (5) are all equivalent statements.
That (1) implies (2) follows from the fact that E(R) is a von Neumann regular ring together with the appropriate applications to Proposition 2.3, Proposition 2.4, and Theorem 2.8. Clearly (4) implies (1).
We now turn to the embedding R → M(R). We begin by recalling the most useful characterization of a flat epimorphism (see) [20]. Proposition 4.7 strengthens the fact that a flat epimorphism is an m-extension (see the remarks after Definition 3 of) [16].

Theorem 4.7. The extension R → S is a flat epimorphism if and only for all s ∈ S there exists r1,...,rn ∈ R such that ri s ∈ R and r1 S +···+ rn S = S.

Proposition 4.8. If the extension R → S is a flat epimorphism, then it is a quasi-rigid extension. P. Bhattacharjee *et al.* / Topology and its Applications 158 (2011) 1802–1814 1813

Proof. The second statement follows from the first statement and Lemma 2.15. As for the first let b ∈ S and choose r1,...,rn ∈ R for which r1 S + ⋯ + rn S = S and rkb ∈ R for each k = 1,...,n. Choose s1,..., sn ∈ S for which 1 = r1s1 +⋯+ rnsn. We leave it to the interested reader to check that AnnS (b) = AnnS (r1b,...,rnb).

Remark 4.9. Notice that the above proof carries through if we just assume that r1 S +⋯+ rn S is a dense ideal of S.

Corollary 4.10. For any reduced ring R, the extension R → M(R) is a quasi rigid extension.
We then provide a new definition for when M(R) is von Neumann regular. Quentel [18] and Olivier [15] were the first to show that conditions (5) and (6) are equal.

Theorem 4.11. Let R be a reduced commutative ring with identity. Then the following statements are equivalent.
(1) R → E(R) is a quasi rigid extension.
(2) R → E(R) is an r-extension.
(3) R → E(R) is an m-extension.
(4) M(R) = E(R).
(5) M(R) is a von Neumann regular ring. (6) Min(R) is compact.

Proof. A quasi rigid extension is a quasi r-extension, according to Proposition 2.3. A quasi rigid extension is an r-extension, according to Theorem 2.8. As a result, (1) implies (2). The content of Lemma 2.7 is that an r-extension is an m-extension. As a result, (2) implies (3).
Suppose R → E(R) is an m-extension. E(R) is von Neumann regular, and thus R → E(R) is an m-extension if and only if E(R) is a flat R-module (see Proposition 1.14 of [12]). Therefore, E(R) is a flat (epimorphic) extension of R. It follows that M(R) = E(R). This proves that (3) implies (4). If M(R) = E(R), then since E(R) is von Neumann regular so is M(R). Hence (4) implies (5).
Suppose M(R) is a von Neumann regular ring. Corollary 4.10 states that R → M(R) is a quasi rigid extension. Hence, by applying Proposition 2.3 and Theorem 2.8, we conclude that Min(R) and Min(M(R)) are homeomorphic. Since M(R) is von Neumann, Min(M(R)) is compact and hence so is Min(R). Therefore, (5) implies (6).
Finally, suppose Min(R) is compact. Then R → Q (R) is an m-extension. It follows that R → E(R) is an m-extension and so E(R) = M(R) is a quasi-rigid extension of R. Consequently, (6) implies (1).

Remark 4.12. Theorem 2.3 of [4] states that for any reduced ring R, the extension R → E(R) is an exoteric homomorphism. It follows by Theorem 4.11 that if Min(R) is not compact, then R → E(R) is an exoteric homomorphism which is not an m-extension.
At this point we are unable to characterize when R → M(R) is a rigid extension. If R satisfies the a.c. then by Proposition 2.4 and Corollary 4.10 it is a rigid extension. However, there are cases when q(R) = M(R), and hence it is a rigid extension, without having R satisfy the a.c. We wonder whether R → M(R) is a rigid extension if and only if q(R) = M(R).

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5. References

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