Systems of Fredholm Integral Equations of Second Kind

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Abstract
Systems of nonlinear integral Fredholm equations of the second kind have been developed. We have reviewed some of the different solutions to solve these systems and the aim of which is to classify and evaluate the chosen method. We stress the importance of this study is coming from the solving of systems of nonlinear Fredholm integral equations of the second kind. And we found that the direct computation methods and the modified Adomian method work effectively for dealing with systems of Fredholm integral equations of the second kind.

Keywords: Systems, linear Fredholm Integral Equations, Nonlinear Fredholm Integral Equations, Second type

1. Introduction

Solving a system of nonlinear equations is a problem that is avoided as much as possible. Numerical solutions have been developed to solve systems of nonlinear Fredholm integral equations of the second type and it was formed in several ways, including direct arithmetic method and modified adomian method, which work effectively to deal with systems of nonlinear fredholm integral equations of the second kind.

It was found that the direct computation method gives the solution accurately and not in the form of a string, and this method is applied to degenerate or separable kernels, and either the modified Adomian method is used repeatedly and comprehensively in this paper. This method can be used in its standard form or combined with conditions phenomenon. After identifying these components, sequential solutions and microsolutions are easily obtained.

2. Systems of Fredholm Integral Equations

We concern ourselves on solving system of fredholm integral equations of the second kind with only two unknown functions $u(x)$ and $v(x)$. Theanalysis can be extended to more than two unknown functions. The standard system offredholm integral equations of the second kind is given by

$$u(x) = f_1(x) + \int_a^b \left(K_1(x,t)u(t) + \tilde{K}_1(x,t)v(t)\right)dt$$
$$v(x) = f_2(x) + \int_a^b \left(K_2(x,t)u(t) + \tilde{K}_2(x,t)v(t)\right)dt \tag{1.1}$$

Where the unknown functions that be determined are $u(x)$ and $v(x)$. Recall that forthe second kind, the unknown functions appear inside and outside the integral sign. Thekernels $K_i(x,t)$, and, and the function$f_1(x)$ and $f_2(x)$ are prescribed realvalued functions.

Recall that we applied the direct computation method and the adomiandecomposition method, together with its related modification, for handling fredholmintegral equations. The aforementioned methods are now well known, hence we select two distinct examples that be examined by using the two methods [1].

Example 2.1
Solve the following system of Fredholm integral equations by using thedirect computation method [1].

$$u(x) = x^2 + \sin x - \frac{\pi^3}{12}x + \int_0^{\frac{\pi}{2}} \big(xu(t) + xv(t)\big)\,dt,$$
$$v(x) = x^2 - \cos x - 2x + \int_0^{\frac{\pi}{2}} \big(xu(t) - xv(t)\big)\,dt, \tag{1.2}$$

Following the analysis presented above, this system can be rewritten as

$$u(x) = x^2 + \sin x + \left(\alpha - \frac{\pi^3}{12}\right)x,$$
$$v(x) = x^2 - \cos x + (\beta - 2)x, \tag{1.3}$$

Where

$$\alpha = \int_0^{\frac{\pi}{2}} \big(u(t) + v(t)\big)\,dt,$$
$$\beta = \int_0^{\frac{\pi}{2}} \big(u(t) - v(t)\big)\,dt, \tag{1.4}$$

Substitute (1.3) into (1.4) and solving the resulting equations we find

$$\alpha = \tfrac{\pi^3}{12}, \beta = 2 \tag{1.5}$$

This in turn gives the exact solutions

$$\big(u(x), v(x)\big) = (x^2 + sin\,x\,, x^2 - cos\,x) \tag{1.6}$$

3. System of Fredholm Integral Equations of Second Kind

We solve the system of fredholm integral equation of second kind [3].

$$\emptyset_i(x) - \lambda \int_a^b \sum_{j=1}^n K_{ij}(x,y)\emptyset_j(y)dy = f_i(x),\ i = 1,2,\dots,n \tag{1.7}$$

Where the kernels $K_{ij}(x,y)$are square-integrable. We first extend the basic interval from$[a,b]to[a, a + n(b - a)]$, and set

$$x + (i-1)(b-a) = X < a + i(b-a),$$
$$y + (j-1)(b-a) = Y < a + j(b-a), \tag{1.8}$$
$$\emptyset(X) = \emptyset_i(x), K(X,Y) = K_{ij}(x,y), f(X) = f_i(x) \tag{1.9}$$

We then obtain the fredholm integral equation of the second kind

$$\emptyset(X) - \lambda \int_a^{a+n(b-a)} K(X,Y)\emptyset(Y)dY = f(X) \tag{1.10}$$

Where the kernel $K(X,Y)$ is discontinuous in general but is square-integrable on account of the square-integrability of $K_{ij}(x,y)$. The solution $\emptyset(X)$ to Eq. (1.10) provides the solutions $\emptyset_i(x)$ to Eq. (1.7) with Equations. (1.8) and (1.9).

4. Numerical Methods for System of Linear Fredholm Integral Equations of Second Kind
Consider the system of linear Fredholm integral equations of second kind of the following form

$$\sum_{j=1}^{n} y_j(x) = f_i(x) + \sum_{j=1}^{n} \int_0^1 K_{ij}(x,t) y_j(t)dt, i = 1,2,\dots,n \quad (1.11)$$

Where $f_i(x)$ and $K_{ij}(x,t)$ are known functions and $y_j(x)$ are the unknown functions for $i, j = 1, 2, \dots, n$.
4.1. Application of haar Wavelet Method [9]. In this section, an efficient algorithm for solving fredholm integral equations with haar wavelets is analyzed. The present algorithm takes the following essential strategy. The haar wavelet is first used to decompose integral equations into algebraic systems of linear equations, which are then solved by collocation methods [4].

4.1.1. haar Wavelets. The compact set of scale functions is chosen as

$$h_0 = \begin{cases} 1, 0 \le x < 1, \\ 0, \text{others} \end{cases} \quad (1.12)$$

The mother wavelet function is defined as

$$h_1(x) = \begin{cases} 1, 0 \le x < \frac{1}{2} \\ -1, \frac{1}{2} \le x < 1, \\ 0, \text{others} \end{cases} \quad (1.13)$$

The family of wavelet functions generated by translation and
Dilation of $h_1(x)$ are given by

$$h_n(x) = h_1(2^j x - k), \quad (1.14)$$

Where $n = 2^j + k, j \ge 0, 0 \le k < 2^j$.

Mutual orthogonalities of all haar wavelets can beexpressed as [4].

$$\int_0^1 h_m(x) h_n(x) dx = 2^{-j} \delta_{mm} = \begin{cases} 2^{-j}, m = n = 2^j + k \\ 0, m \ne n \end{cases} \quad (1.15)$$

5. Systems of Nonlinear Fredholm Integral Equations
We study systems of nonlinear fredholm integral
Equations of the second kind given by

$$\begin{aligned} u(x) &= f_1(x) + \int_a^b (K_1(x,t) F_1(u(t)) + \tilde{K}_1(x,t) \tilde{F}_1(v(t)))dt \\ v(x) &= f_2(x) + \int_a^b (K_2(x,t) F_2(u(t)) + \tilde{K}_2(x,t) \tilde{F}_2(v(t)))dt \end{aligned} \quad (1.16)$$

The unknown functions $u(x)$ and $v(x)$, that be determined, occur inside and outside the integral sign. The kernels $K_i(x,t)$ and $\tilde{K}_i(x,t)$and the function $f_i(x)$ are given real-valued functions, for $i = 1, 2$. The functions F_i and $\tilde{F}_i$, for $i = 1, 2$ are nonlinear functions of $u(x)$ and $v(x)$.
In this paper, the system of nonlinear fredholm integral equations can be handled by four distinct methods, namely the direct computation method, the modified adomian method, the successive approximations method, and the series solution method. Although the aforementioned methods work effectively for handling the systems of nonlinear fredholm integral equations, but only the first two methods be used in this section [2].

5.1 The Direct Computation Method
The direct computation method be applied to solve the systems of nonlinear Fredholm integral equations of the second kind. The method approaches Fredholm integral equations in a direct manner and gives the solution in an exact form and not in a series form. In what follows we summarize the necessary steps needed to apply this method. The method be applied for the degenerate or separable kernels of the form

$$\begin{aligned} K_1(x,t) &= \sum_{k=1}^{n} g_k(x) h_k(t), \tilde{K}_1(x,t) = \sum_{k=1}^{n} \tilde{g}_k(x) \tilde{h}_k(t) \\ K_2(x,t) &= \sum_{k=1}^{n} r_k(x) s_k(t), \tilde{K}_2(x,t) = \sum_{k=1}^{n} \tilde{r}_k(x) \tilde{s}_k(t) \end{aligned} \quad (1.17)$$

The direct computation method can be applied as follows:
1. We first substitute (1.17) into the system (1.16) to obtain

$$u(x) = f_1(x) + \sum_{k=1}^{n} g_k(x) \int_a^b h_k(t) F_1(u(t))dt + \sum_{k=1}^{n} \tilde{g}_k(x) \int_a^b \tilde{h}_k(t)\tilde{F}_1(v(t))dt$$
$$v(x) = f_2(x) + \sum_{k=1}^{n} r_k(x) \int_a^b s_k(t) F_2(u(t))dt + \sum_{k=1}^{n} \tilde{r}_k(x) \int_a^b \tilde{s}_k(t)\tilde{F}_2(v(t))dt \quad (1.18)$$

2. Each integral at the right side depends only on the variable $t$with constant limits of integration for t. This means that each integral is equivalent to a constant. Based on this, Equation (1.18) becomes

$$u(x) = f_1(x) + \alpha_1 g_1(x) + \cdots + \alpha_n g_n(x) + \beta_1 \tilde{g}_1(x) + \cdots + \beta_n \tilde{g}_n(x)$$
$$v(x) = f_2(x) + \gamma_1 r_1(x) + \cdots + \gamma_n r_n(x) + \delta_1 \tilde{r}_1(x) + \cdots + \delta_n \tilde{r}_n(x) \quad (1.19)$$

Where

$$\alpha_i = \int_a^b h_i(t) F_1(u(t))dt, 1 \le i \le n$$
$$\beta_i = \int_a^b \tilde{h}_i(t)\tilde{F}_1(v(t))dt, 1 \le i \le n \quad (1.20)$$
$$\gamma_i = \int_a^b s_i(t) F_2(u(t))dt, 1 \le i \le n$$
$$\delta_i = \int_a^b \tilde{s}_i(t)\tilde{F}_2(v(t))dt, 1 \le i \le n$$

3. Substituting (1.19) into (1.20) gives a system of $n$algebraic equations that can be solved to determine the constants α_i,β_i, γ_i, andδ_i. To facilitate the computational work, we can use the computer symbolic systems such as Maple and Mathematica. Using the obtained numerical values of these constants into (1.19), the solutions $u(x)$ and $v(x)$ of the system of nonlinear fredholm integral equations (1.16) follow immediately. The analysis presented above can be explained by studying the following examples [2].

Example 5.1.1

Solve the following system of nonlinear Fredholm integral equations by using the direct computation method [2].

$$u(x) = \sin x + (1 - 2\pi)\cos x + \int_0^{\pi} \cos x(u^2(t) + v^2(t))dt$$
$$v(x) = \sin x + \cos x + \int_0^{\pi} (u^2(t) + v^2(t))dt \quad (1.20)$$

This system can be rewritten as

$$u(x) = \sin x + (1 - 2\pi + \alpha + \beta)\cos x$$
$$v(x) = \sin x + \cos x + (\alpha - \beta) \quad (1.21)$$

Where

$$\alpha = \int_0^{\pi} u^2(t)dt, \beta = \int_0^{\pi} v^2(t)dt \quad (1.22)$$

To determineα, andβ, we substitute (1.21) into (1.22), and solving the resulting system, we obtain

$$\alpha = \pi, \beta = \pi \quad (1.23)$$

Substituting this result into (1.21) leads to the exact solutions

$$(u(x), v(x)) = (\sin x + \cos x, \sin x - \cos x) \quad (1.24)$$

Example 5.1.2

Solve the following system of nonlinear fredholm integral equations by using the direct computation method [2].

$$u(x) = 1 - 6x + \ln x + \int_{0^+}^{1} x(u^2(t) + v^2(t))dt$$
$$v(x) = 1 + 4x^2 - \ln x + \int_{0^+}^{1} x(u^2(t) - v^2(t))dt \quad (1.25)$$

Proceeding as before, this system can be rewritten as

$$u(x) = 1 + (\alpha + \beta - 6)x + \ln x$$
$$v(x) = 1 + (4 + \alpha - \beta)x^2 - \ln x \quad (1.26)$$

Where

$$\alpha = \int_{0^+}^{1} u^2(t)dt, \beta = \int_{0^+}^{1} v^2(t)dt \quad (1.27)$$

To determine α, and β, we substitute (1.26) into (1.27), and proceeding as before to obtain

$$\alpha = 1, \beta = 5 \quad (1.28)$$

This in turn gives the exact solutions

$$\big(u(x), v(x)\big) = (1 + \ln x, 1 - \ln x) \quad (1.29)$$

5.2 The Modified Adomian Decomposition Method

The modified Adomian decomposition method [11–12] was frequently and thoroughly used in this text. The method decomposes the linear terms $u(x)$ and $v(x)$ by an infinite sum of components of the form

$$u(x) = \sum_{n=0}^{\infty} u_n(x), v(x) = \sum_{n=0}^{\infty} v_n(x) \quad (1.30)$$

Where the components $u_n(x)$ and $v_n(x)$ be determined recurrently. The method can be used in its standard form, or combined with the noise terms phenomenon.
However, the nonlinear functions F_i and $\tilde{F}_i$, for $i = 1,2$, in (1.16) should be replaced by the adomian polynomials A_n defined by

$$A_n = \frac{1}{n!}\frac{d^n}{d\lambda^n}\Big[F\Big(\sum_{i=0}^{n} \lambda^i u_i\Big)\Big]_{\lambda=0}, n = 0,1,2, \ldots \quad (1.31)$$

Substituting the aforementioned assumptions for the linear and the nonlinear terms into the system (1.16) and using the recurrence relations we can determine the components $u_n(x)$ and $v_n(x)$. Having determined these components, the series solutions and the exact solutions are readily obtained [2].

Example 5.2.1

Use the modified adomian decomposition method to solve the following system of nonlinear Fredholm integral equations [2].

$$u(x) = \sin x - \pi + \int_0^{\pi} \big((1 + xt)u^2(t) + (1 - xt)v^2(t)\big)\, dt$$
$$v(x) = \cos x + \frac{\pi^2}{2}x + \int_0^{\pi} \big((1 - xt)u^2(t) - (1 + xt)v^2(t)\big)\, dt \quad (1.32)$$

Substituting the linear terms $u(x)$ and $v(x)$ and the nonlinear terms $u^2(t)$ and $v^2(t)$ from (1.30) and (1.31) respectively into (1.22) gives

$$\sum_{n=0}^{\infty} u_n(x) = sin\, x - \pi + \int_0^{\pi} \left((1 + xt)\sum_{n=0}^{\infty} A_n(t) + (1 - xt)\sum_{n=0}^{\infty} B_n(t) \right) dt$$
$$\sum_{n=0}^{\infty} v_n(x) = cos\, x + \frac{\pi^2}{2}x + \int_0^{\pi} \left((1 - xt)\sum_{n=0}^{\infty} A_n(t) - (1 + xt)\sum_{n=0}^{\infty} B_n(t) \right) dt$$
$$(1.33)$$

The modified decomposition method be used here, hence we set there cursive relation

$$u_0(x) = \sin x \ , v_0 = \cos x$$
$$u_1(x) = -\pi + \int_0^{\pi} \left((1+xt)u_0^2(t) + (1-xt)v_0^2(t)\right) dt = 0 \tag{1.34}$$
$$v_1(x) = \frac{\pi^2}{2}x + \int_0^{\pi} \left((1-xt)u_0^2(t) - (1+xt)v_0^2(t)\right) dt = 0$$

This in turn gives the exact solutions

$$(u(x), v(x)) = (\sin x \, , \cos x) \tag{1.35}$$

Example 5.2.2
Use the modified adomian decomposition method to solve the following system of nonlinear fredholm integral equation [2].

$$u(x) = e^x + e^{x+1} + \int_0^1 (e^{x-3t}v^2(t) + e^{x-6t}w^2(t))dt,$$
$$v(x) = e^{2x} - 2e^x + \int_0^1 (e^{x-6t}w^2(t) + e^{x-2t}u^2(t))dt, \tag{1.36}$$
$$w(x) = e^{3x} - 2e^x + \int_0^1 (e^{x-2t}u^2(t) + e^{x-4t}v^2(t))dt,$$

Using the modified decomposition method, we set the recurrence relation

$$u_0(x) = e^x, v_0(x) = e^{2x}, w_0(x) = e^{3x}$$
$$u_1(x) = -e^{x+1} + \int_0^1 (e^{x-3t}v_0^2(t) + e^{x-6t}w_0^2(t))dt = 0, \tag{1.37}$$
$$v_1(x) = -2e^x + \int_0^1 (e^{x-6t}w_0^2(t) + e^{x-2t}u_0^2(t))dt = 0,$$
$$w_1(x) = -2e^x + \int_0^1 (e^{x-2t}u^2(t) + e^{x-4t}v^2(t))dt = 0,$$

Consequently, the exact solutions are given by

$$(u(x), v(x), w(x)) = (e^x, e^{2x}, e^{3x}) \tag{1.38}$$

Results
Within our solving for the systems of Fredholm integral equations of the second kind we found the following some results : These solutions moved away from the purely theoretical aspect and provided practical examples without compromising scientific accuracy so that the information would be easy,the direct computation method and the modified Adomian method work effectively to deal with these systems, the direct computation method gives the solution accurately and its applied to degenerate or separable kernels, the modified Adomian method is used repeatedly and comprehensively to obtain sequential solutions.

Conclusion:
We discussed some important methode for solving the systems of nonlinear Fredholm integral equations of the second kindse. These equations include some methods for solving them, namely, direct arithmetic method, modified Adomian method, successive approximation method, and series solution method. From these methods, we used the direct calculation method which gives the solution accurately, not in the form of a series and the modified Adomian method which gives series solutions and exact solutions easily.

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