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On the second degree equation with three unknowns $5x^2 + 4y^2 = 189z^2$

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Abstract

The homogeneous ternary quadratic equation given by $5x^2 + 4y^2 = 189z^2$ is analysed for its integral points on it. Also, formulae for generating sequence of integer solutions based on the given solution are presented.

Keywords: Ternary quadratic, Integer solutions, Homogeneous

Introduction

It is well known that the quadratic Diophantine equations with three unknowns (homogeneous or non-homogeneous) are rich in variety [1, 2, 3]. In particular, one may refer [4-17] for homogeneous or non-homogeneous ternary quadratic Diophantine equations that are analysed for obtaining their corresponding non-zero distinct integer solutions. In this communication, yet another interesting homogeneous ternary quadratic Diophantine equation given by $5x^2 + 4y^2 = 189z^2$ is analysed for its non-zero distinct integer solutions through different methods. A few interesting properties between the solutions are presented. Also, formulas for generating sequence of integer solutions based on the given solution are presented.

Methods of Analysis

The ternary quadratic equation to be solved for its integer solutions is

$$5x^2 + 4y^2 = 189z^2 \tag{1}$$

Introduction of the linear transformations

$$x = 6X,\ y = 3Y,\ z = 2W \tag{2}$$

In (1) leads to

$$Y^2 + 5X^2 = 21W^2 \tag{3}$$

We present below different methods of solving (3) and in view of (2), one obtains different sets of integer solutions to (1).

Method: 1
Is written in the ratio form as

$$\frac{Y+4W}{W+X}=\frac{5(W-X)}{Y-4W}=\frac{\alpha}{\beta},\beta\neq 0 \tag{4}$$

Which is equivalent to the system of double equations

$$\alpha X-\beta Y+(\alpha-4\beta)W=0$$
$$5\beta X+\alpha Y-(4\alpha+5\beta)W=0$$

Applying the method of cross-multiplication to the above system of equations, note that

$$X=8\alpha\beta+5\beta^2-\alpha^2$$
$$Y=10\alpha\beta-20\beta^2+4\alpha^2$$
$$W=\alpha^2+5\beta^2$$

In view of (2), one has

$$x=6(8\alpha\beta+5\beta^2-\alpha^2)$$
$$y=3(10\alpha\beta-20\beta^2+4\alpha^2)$$
$$z=2(\alpha^2+5\beta^2)$$

Which satisfy (1)

Note: 1
Expressing (3) in the ratio forms as below

$$\frac{Y+4W}{5(W+X)}=\frac{W-X}{Y-4W}=\frac{\alpha}{\beta}$$

$$\frac{Y+4W}{W-X}=\frac{5(W+X)}{Y-4W}=\frac{\alpha}{\beta}$$

$$\frac{Y+4W}{5(W-X)}=\frac{W+X}{Y-4W}=\frac{\alpha}{\beta}$$

One can also obtain other solutions to (1) by using the above method.

Method: 2
(3) Can be written as

$$Y^2=21W^2-5X^2 \tag{5}$$

Introducing the linear transformations

$$W=\overline{X}+5T,\ X=\overline{X}+21T,\ Y=4S \tag{6}$$

In (5), it is written as

$$\overline{X}^2=S^2+105T^2 \tag{7}$$

Which is satisfied by

$$T = 2rs, S = 105r^2 - s^2, \overline{X} = 105r^2 + s^2$$

Substituting the above values in (6) and using (2), the corresponding integer solutions to (1) are given by

$$x=6(105r^2 + s^2 + 42rs),\ y= 12(105r^2 - s^2),\ z= 2(105r^2 + s^2 + 10rs)$$

Method: 3
(7) Is written as the system of double equations in Table 1 as follows.

Table 1: System of Double Equations

System	1	2	3	4	5	6	7	8	9	10
$\overline{X}+S$	$105T^2$	$35T^2$	$21T^2$	$7T^2$	$3T^2$	T^2	105T	35T	21T	15T
$\overline{X}-S$	1	3	5	15	35	105	T	T	5T	7T

Solving each of the above system of double equations,the values of $\overline{X}, S$ *and* T are obtained. Using these values in (6) and employing (2) the value of $x, y \& z$ satisfying (1) are obtained. For simplicity and brevity, in what follows, the integer solutions thus obtained are exhibited.

Solutions for system: 1

$$x= 6(210K^2 + 252K + 74),\ y= 12(210K^2 + 210K + 52),\ z= 2(210K^2 + 220K + 58)$$

Solutions for system: 2

$$x= 6(70K^2 + 112K + 40),\ y= 12(70K^2 + 70K + 16),\ z= 2(70K^2 + 80K + 24)$$

Solution for system: 3

$$x= 6(42K^2 + 84K + 34),\ y= 12(42K^2 + 42K + 8),\ z= 2(42K^2 + 52K + 18)$$

Solution for system: 4

$$x= 6(14K^2 + 56K + 32),\ y= 12(14K^2 + 14K - 4),\ z= 2(14K^2 + 24K + 16)$$

Solution for system: 5

$$x= 6(6K^2 + 48K + 40),\ y= 12(6K^2 + 6K - 16),\ z= 2(6K^2 + 16K + 24)$$

Solution for system: 6

$$x= 6(2K^2 + 44K + 74),\ y= 12(2K^2 + 2K - 52),\ z= 2(2K^2 + 12K + 58)$$

Solution for system: 7

x= 444T, y= 624T, z= 116T

Solution for system: 8

x= 234T, y= 204T, z= 46T

Solution for system: 9

x= 204T, y= 96T, z= 36T

Solution for system: 10

x= 192T, y= 48T, z= 32T

Method: 4

(3) Is written as

$$21W^2 - 5X^2 = Y^2 * 1 \tag{8}$$

Assume Y as

$$Y = 21a^2 - 5b^2 \tag{9}$$

Write 1 on the R.H.S. of (8) as

$$1= \frac{(\sqrt{21}+\sqrt{5})(\sqrt{21}-\sqrt{5})}{16} \tag{10}$$

Using (9) & (10) in (8) and employing the method of factorization, consider

$$\sqrt{21}W + \sqrt{5}X = \frac{1}{4}(\sqrt{21}+\sqrt{5})(\sqrt{21}a+\sqrt{5}b)^2$$

After Equating the corresponding terms on both sides , it is seen that

$$W= \frac{1}{4}(21a^2 + 5b^2 + 10ab), \quad X= \frac{1}{4}(21a^2 + 5b^2 + 42ab) \tag{11}$$

Using (9) and (11) in (2), one has

$$x= \frac{3}{2}(21a^2 + 5b^2 + 42ab), \quad y= 3(21a^2 - 5b^2), \; z = \frac{1}{2}(21a^2 + 5b^2 + 10ab) \tag{12}$$

As our interest is on finding integer solutions, the values of a and b should be of the same parity.

Note: 2

In addition to (10), the integer 1 on the R.H.S. of (8) is written as

$$1= (\sqrt{21}+2\sqrt{5})*(\sqrt{21}-2\sqrt{5})$$

Following the above procedure, one may obtain different set of integer solutions to (1).

Method 5:

In (3), assume W as

$$W= a^2 + 5b^2 \tag{13}$$

Write 21 as

$$21 = (1+i\,2\sqrt{5})(1-i\,2\sqrt{5}) \tag{14}$$

Using (13) and (14) in (3) and employing the method of factorisation, consider

$$Y + i\sqrt{5}X = (1+i\,2\sqrt{5})(a+i\sqrt{5}b)^2$$

Equating the real and imaginary parts, one has

$$Y= a^2 - 5b^2 - 20ab, \; X= 2(a^2 - 5b^2) + 2ab$$

Therefore, in view of (2), the corresponding integer solutions to (1) are given by

$$x = 12\left(a^2 - 5b^2 + ab\right),\ y = 3\left(a^2 - 5b^2 - 20ab\right),\ z = 2\left(a^2 + 5b^2\right)$$

Note 3
In addition to (14), the integer 21 on the R.H.S. of (3) is written as

$$21 = (4 + i\sqrt{5})(4 - i\sqrt{5})$$

Following the above procedure, one may obtain different set of integer solutions to (1).

Method 6
(3) Is written as

$$Y^2 + 5X^2 = 21W^2 * 1 \tag{15}$$

Write the integer 1 on the R.H.S. of (15) as

$$1 = \frac{1}{81}\left(1 + i4\sqrt{5}\right)\left(1 - i4\sqrt{5}\right) \tag{16}$$

Using (13), (14) and (16) in (15) and employing the method of factorisation, consider

$$Y + i\sqrt{5}X = \left(1 + i2\sqrt{5}\right)\left(a^2 - 5b^2 + i\sqrt{5}\,2ab\right)\frac{\left(1 + i4\sqrt{5}\right)}{9}$$

Equating the real and imaginary parts,

$$Y = \frac{1}{3}\left[\left(a^2 - 5b^2\right)(-13) - 20ab\right]$$

$$X = \frac{1}{3}\left[2\left(a^2 - 5b^2\right) - 26ab\right]$$

In view of (2), the corresponding integer solutions to (1) are given by

$$x = 4\left(a^2 - 5b^2\right) - 52ab$$

$$y = -13\left(a^2 - 5b^2\right) - 20ab$$

$$z = 2\left(a^2 + 5b^2\right)$$

Note 4
In addition to (16), the integer 1 on the R.H.S. of (15) is written as

$$1 = \frac{\left(2 + i3\sqrt{5}\right)\left(2 - i3\sqrt{5}\right)}{49}$$

Following the above procedure, one may obtain different set of integer solution to (1).
It is worth mentioning that the above solutions are different from [17].

Generation of Solutions
Different formulas for generating sequence of integer solutions based on the given solutions are presented below:
Let (x_0, y_0, z_0) be any given solutions to (1)

Formula 1

Let (x_1, y_1, z_1) given by

$$x_1 = 7x_0, y_1 = -7y_0 + 7h, z_1 = 7z_0 + h \tag{17}$$

Be the second solution to (1). Using (17) in (1) and simplifying, one obtains

h= $56\,y_0 + 378z_0$

In view of (17), the values of $y_1 and\ z_1$ are written in the matrix form as

$$(y_1, z_1)^t = M(y_0, z_0)^t$$

Where

$$M=\begin{bmatrix} 385 & 2646 \\ 56 & 385 \end{bmatrix}$$

And t is the transpose.

The repetition of the above process leads to the n^{th} solutions y_n, z_n given by

$$(y_n, z_n)^t = M^n (y_0, z_0)^t$$

If α, β are the distinct eigen values of M, then

$$\alpha = 763, \beta = 7$$

We know that

$$M^n = \frac{\alpha^n}{(\alpha - \beta)}(M - \beta I) + \frac{\beta^n}{\beta - \alpha}(M - \alpha I), I = 2 \times 2 \ Identity\,Matrix$$

Thus, the general formulas for integer solutions to (1) are given by

$$x_n = 7^n x_0,$$

$$\begin{pmatrix} y_n \\ z_n \end{pmatrix} = \frac{1}{108}\begin{bmatrix} 54(\alpha^n + \beta^n) & 378(\alpha^n - \beta^n) \\ 8(\alpha^n - \beta^n) & 54(\alpha^n + \beta^n) \end{bmatrix}\begin{pmatrix} y_0 \\ z_0 \end{pmatrix}$$

Formula 2

Let (x_1, y_1, z_1) given by

$$x_1 = 9x_0 + 6h, y_1 = 9y_0, z_1 = -9z_0 + h \tag{18}$$

Be the second solution to (1). Using (18) in (1) and simplifying, one obtains

h= $60x_0 + 378z_0$

In view of (17), the values of $x_1 and\ z_1$ are written in the matrix form as

$$(x_1, z_1)^t = M(x_0, z_0)^t$$

Where

$$M=\begin{bmatrix}369 & 2268\\ 60 & 369\end{bmatrix}$$

and t is the transpose.

The repetition of the above process leads to the n^{th} solutions x_n, z_n given by

$$(x_n, z_n)^t = M^n (x_0, z_0)^t$$

If α, β are the distinct eigen values of M, then

$$\alpha = 729, \beta = 9$$

We know that

$$M^n = \frac{\alpha^n}{(\alpha-\beta)}(M-\beta I)+\frac{\beta^n}{\beta-\alpha}(M-\alpha I), I = 2\times 2 \ Identity\, Matrix$$

Thus, the general formulas for integer solutions to (1) are given by

$$\begin{pmatrix}x_n\\ z_n\end{pmatrix}=\frac{1}{180}\begin{bmatrix}90(\alpha^n+\beta^n) & 567(\alpha^n-\beta^n)\\ 15(\alpha^n-\beta^n) & 90(\alpha^n+\beta^n)\end{bmatrix}\begin{pmatrix}x_0\\ z_0\end{pmatrix}$$

$$y_n = 9^n y_0$$

Formula 3

Let (x_1, y_1, z_1) given by

$$x_1 = h - 9x_0, y_1 = -9y_0 + h, z_1 = 9z_0 \quad (19)$$

Be the second solution to (1). Using (19) in (1) and simplifying, one obtains

$$h = 10x_0 + 8y_0$$

In view of (17), the values of $x_1 and\ y_1$ are written in the matrix form as

$$(x_1, y_1)^t = M(x_0, y_0)^t$$

Where

$$M=\begin{bmatrix}1 & 8\\ 10 & -1\end{bmatrix}$$

And t is the transpose.

The repetition of the above process leads to the n^{th} solutions x_n, y_n given by

$$(x_n, y_n)^t = M^n (x_0, y_0)^t$$

If α, β are the distinct eigen values of M, then

$$\alpha = 9, \beta = -9$$

We know that

$$M^n = \frac{\alpha^n}{(\alpha-\beta)}(M-\beta I)+\frac{\beta^n}{\beta-\alpha}(M-\alpha I), I = 2\times 2\ Identity\,Matrix$$

Thus, the general formulas for integer solutions to (1) are given by

$$\begin{pmatrix} x_n \\ y_n \end{pmatrix} = \frac{1}{9}\begin{bmatrix} 5\alpha^n + 4\beta^n & 4(\alpha^n - \beta^n) \\ 5(\alpha^n - \beta^n) & 4\alpha^n + 5\beta^n \end{bmatrix}\begin{pmatrix} y_0 \\ z_0 \end{pmatrix}$$

$$z_n = 9^n z_0,$$

Conclusion:

In this paper, an attempt has been made to obtain non-zero distinct integer solutions to the ternary quadratic Diophantine equation $5x^2+4y^2=189z^2$ representing homogeneous cone. As there are varieties of cones, the readers may search for other forms of cones to obtain integer solutions for the corresponding cones.

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